A TREATISE

ON

DIFFERENTIAL EQUATIONS,

AND ON THE

CALCULUS OF FINITE DIFFERENCES.

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CONTENTS.

DIFFERENTIAL EQUATIONS.

SECTION I.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND DEGREE.

SECTION II.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER, BUT NOT OF THE FIRST DEGREE.

SECTION III.

SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS.

SECTION IV.

DIFFERENTIAL EQUATIONS OF THE SECOND ORDER, AND OF HIGHER ORDERS.

SECTION V.

DIFFERENTIAL EQUATIONS INVOLVING TWO OR MORE INDEPENDENT VARIABLES.

FINITE DIFFERENCES.

SECTION I.

DIRECT METHOD OF DIFFERENCES.

SECTION II.

INVERSE METHOD OF DIFFERENCES.

SECTION III.

EQUATIONS OF DIFFERENCES.

SECTION IV.

SUMMATION OF SERIES.

Students reading this work for the first time may confine their attention to the following Articles:

DIFFERENTIAL EQUATIONS.

1—4, 6—10, 11, 13—15, 24, 29, 30, 34, 37, 45, 47, 49—51, 54, 55, 74—81.

FINITE DIFFERENCES.

1—16, 18—20, 23, 40—45, 47, 49—51, 57, 58, 72—74, 88, 105—107.

ERRATA.

Page	Line	Error.	Correction.
63	9	$C'e^{hx+kx\sqrt{-1}}$	$C'e^{hx-kx\sqrt{-1}}$
27	2 from bottom	given	giving.

FINITE DIFFERENCES.

Page	Line	Error.	Correction.
54	5	u^2_{x+n}	${}^2u_{x+n}$.
81	3	$c_1' \cos x\theta$	$c_1' \sin x\theta$
101	3	Art. 112.	Art. 114.

ERRATA.

DIFFERENTIAL EQUATIONS.

P. 37, line 3 from bottom, *for* $d_p \phi(p)$ *read* $d_p f(p)$.

46, 6 from bottom, *for* $y=f(x, c)$ *read* $y=F(x, c)$.

55, 5 from bottom, *for* $y d_x y$ *read* $y d_x^2 y$.

56, 6 *for* $(1+p^2)^{\frac{1}{2}}$ *read* $(1+p^2)^{\frac{3}{2}}$.

73, 2, 9, 16 from bottom, *for* $=1$ *read* $=(-1)^{n-1}$; and line 1, 2, *for* $d_{m=a_1}$ *read* $d_{m=-a_1}$.

74, 1, *for* $d_{m=a}$ *read* $d_{m=-a}$; and line 10, *for* $d_{m=a} M$ *read* $-d_{m=-a} M$; also line 12, *for* y *read* $-y$.

75, 2, *for* $d_{m=a} M$ *read* $d_{m=-a} M$; and line 4, *for* y *read* $-y$.

FINITE DIFFERENCES.

P. 23, line 3, *for* being 1 *read* being $f(1)$.

51, 15, *for* a^x *read* a_x.

54, 1, *for* these values *read* this value.

60, 3 from bottom, *for* $=X$, *read* $=a^2 X$.

19, 9, instead of the paragraph beginning $\Delta(u_x v_x) = \Delta u_x \,.\, v_x + \&c.$

read $\Delta(u_x v_x) = \Delta u_x \,.\, v_{x+1} + \Delta v_x \,.\, u_x = (\Delta + \Delta v^{-1})\, u_x v_{x+1}$,

supposing Δ to affect u_x only, and using Δv^{-1} for an operation affecting v_x only, such that $(\Delta v^{-1})^r v_x = \Delta^r v_{x-r}$;

then $\Delta^2(u_x v_x) = (\Delta + \Delta v^{-1})\, \Delta\, (u_x v_{x+1}) = (\Delta + \Delta v^{-1})^2 u_x v_{x+2}$;

and, generally,

$$\Delta^n(u_x v_x) = (\Delta + \Delta v^{-1})^n u_x v_{x+n},$$

$$= \Delta^n u_x \,.\, v_{x+n} + n\, \Delta^{n-1} u_x\, \Delta v_{x+n-1} + \frac{n(n-1)}{1 \,.\, 2}\, \Delta^{n-2} u_x\, \Delta^2 v_{x+n-2} + \&c.$$

INTEGRATION

OF

DIFFERENTIAL EQUATIONS

BETWEEN TWO OR MORE VARIABLES.

SECTION I.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND DEGREE.

1. In that part of the Integral Calculus which relates to the integration of explicit functions of one variable, we have to determine the relation between y and x from the equation

$$d_x y = f(x);$$

in the present portion, we have to determine it from the equation

$$f(d_x^n y, \quad d_x^{n-1} y, \dots d_x y, \quad y, \quad x) = 0;$$

or to assign the relation between x, y, z (where z is a function of the independent variables x and y), or between a greater number of variables and their functions, from the equation

$$f(d_x z, \quad d_y z, \quad z, \quad x, \quad y) = 0,$$

or from other equations in which a greater number of variables and differential coefficients of higher orders are involved.

2. A differential equation is said to be of the n^{th} order, when the differential coefficient of the highest order which it involves is the n^{th}.

A differential equation of any order is said, moreover, to be of the first, second, &c., degree, when the differential coefficient which marks its order, is raised to the first, second, &c., power.

To integrate a differential equation of any order, is to pass to the primitive equation between the variables and the constants, from which the proposed may have been derived by the process of differentiation.

3. We shall begin with the simplest case, viz. that of differential equations of the first order and degree, which will be of the form

$$M + N d_x y = 0,$$

M and N being functions of x and y.

Every differential equation of the first order and degree is either the direct derived equation of a primitive; or it results from the combination of the derived equation with its primitive, so as to eliminate a constant which enters in each only to the first power; the former sort are called exact, the latter inexact.

4. First, let $u = f(x, y) = 0$ be an equation between x and y, by virtue of which y is a function of x; then, as is proved in the Differential Calculus, $d_x y$ is given by the equation

$$d_{(x)} u + d_{(y)} u \,.\, d_x y = 0,$$

or, since $d_{(x)} u$, $d_{(y)} u$, are functions of x and y which we may represent by M and N,

$$M + N d_x y = 0,$$

a differential equation of the first order and degree, of which $f(x, y) = 0$ is the primitive or integral. Now if $f(x, y)$, besides other constants which are affected with x and y, contain a term $+ C$ independent of x and y; this will not enter into M and N, having disappeared in differentiating; and if there be no such term, we may add it, and $f(x, y) + C = 0$

is still a relation between x and y which satisfies the equation

$$M + Nd_x y = 0;$$

under this form it is called the complete integral; and the constant C, which does not appear in the differential equation, is called the arbitrary constant; if the integral did not contain such a term as $+ C$, it would not be sufficiently general, and would be only a particular case of the complete integral.

We shall presently give the test which every equation of this sort must satisfy, and the mode of integrating it. It is evident that no equation of the first order which is not of the first degree can be exact.

5. Next, let C_1 be another constant which enters to the first power in the equation

$$f(x, y) + C = 0,$$

then C_1 will be affected with x and y, and will consequently appear to the first power in

$$M + Nd_x y = 0;$$

and if a value of C_1 be obtained from either of these equations and substituted in the other, the result will be

$$M_1 + N_1 d_x y = 0,$$

an equation of the first order and degree, involving all the constants which enter into $f(x, y) + C = 0$, except C_1. Hence whilst the direct derived equation of

$$f(x, y) + C = 0, \text{ viz. } M + Nd_x y = 0,$$

does not involve the term C which is independent of x and y, there will be as many other differential equations of the first order and degree that have $f(x, y) + C = 0$ for their primitive, as it has independent constants entering only in the first power; if any constant enter in a dimension above the first, the differential equation obtained by eliminating it, will evidently not be of the first degree.

There are two principal methods of integrating equations of this sort, which consist either in separating the variables, by substitution, or some algebraical process; or in restoring the factor which makes them exact.

Exact Differential Equations of the First Order.

6. Let y be a function of x determined by the equation

$$u = f(x, y) = 0;$$

then the equation which gives the value of $d_x y$ is

$$d_{(x)}u + d_{(y)}u \cdot d_x y = 0, \qquad \text{or} \quad M + N d_x y = 0;$$

the notation $d_{(x)}u$, $d_{(y)}u$, implying that these differential coefficients are formed on the hypothesis that y is not a function of x, i. e. that x and y are independent; then, as proved in the differential Calculus,

$$d_{(y)}M = d_{(x)}N.$$

Conversely, an equation of the form $M + N d_x y = 0$ being proposed in which M and N are functions of x and y, if the condition

$$d_{(y)}M = d_{(x)}N$$

(which is called the criterion of integrability) be satisfied, the equation results from the immediate differentiation of an equation of the form $f(x, y) = 0$; if this condition be not satisfied, there exists no equation by the simple differentiation of which, the given equation can be produced.

7. To integrate the exact differential equation

$$M + N d_x y = 0.$$

Let the equation from which it is derived be

$$u = f(x, y) = 0;$$

then $d_{(x)}u = M$, $d_{(y)}u = N$, and $d_{(y)}M = d_{(x)}N$;

$$\therefore \quad u = \int_{(x)} M + Y,$$

denoting by Y a function of y which may have disappeared, since M is the differential coefficient of u relative to x, on the hypothesis that x and y are independent;

$$\therefore \quad d_{(y)}u = d_{(y)} \left\{ \int_{(x)} M \right\} + d_y Y = N,$$

$$\therefore \quad d_y Y = N - d_{(y)} \left\{ \int_{(x)} M \right\}, \quad \text{and} \quad Y = \int_y \left\{ N - d_{(y)} \left(\int_{(x)} M \right) \right\},$$

$$\therefore \quad u = \int_{(x)} M + \int_y \left\{ N - d_{(y)} \left(\int_{(x)} M \right) \right\} + C = 0,$$

the complete integral involving one arbitrary constant.

8. Obs. The equation $Y = \int_y \left\{ N - d_{(y)} \left(\int_{(x)} M \right) \right\}$ will be absurd, unless the expression $N - d_{(y)} \left(\int_{(x)} M \right)$ be independent of x; therefore its differential coefficient with respect to x must vanish;

$$\therefore \quad d_{(x)}N - d_{(x)}d_{(y)} \left(\int_{(x)} M \right) = d_{(x)}N - d_{(y)}d_{(x)} \left(\int_{(x)} M \right) = d_{(x)}N - d_{(y)}M$$

must equal zero, which it does, since the criterion of integrability is supposed to be satisfied.

9. As the simplest case of exact equations, we may first notice those in which the variables are separated; they will be of the form

$$X + Y d_x y = 0,$$

where X denotes a function of x only, and Y a function of y only; here the criterion of integrability is manifestly satisfied, for

$$d_{(y)}X = d_{(x)}Y = 0;$$

and the complete integral is

$$\int_x X + \int_x Y d_x y = C, \quad \text{or} \quad \int_x X + \int_y Y = C.$$

To this case may likewise be reduced the equation

$$X Y_1 + Y X_1 d_x y = 0,$$

which becomes, when divided by $X_1 Y_1$,

$$\frac{X}{X_1} + \frac{Y}{Y_1} d_x y = 0.$$

Ex. 1. $$\frac{1}{\sqrt{a^2 - x^2}} + \frac{1}{\sqrt{a^2 - y^2}} d_x y = 0;$$

$$\therefore \quad \sin^{-1}\frac{x}{a} + \sin^{-1}\frac{y}{a} = \sin^{-1}\frac{C}{a},$$

$$\text{or} \quad \sin^{-1}\left(\frac{x}{a}\sqrt{1 - \frac{y^2}{a^2}} + \frac{y}{a}\sqrt{1 - \frac{x^2}{a^2}}\right) = \sin^{-1}\frac{C}{a},$$

$$\text{or} \quad x\sqrt{a^2 - y^2} + y\sqrt{a^2 - x^2} = aC;$$

at which we may also arrive, by multiplying the proposed equation by xy, and integrating by parts, which gives

$$-y\sqrt{a^2 - x^2} + \int_x \sqrt{a^2 - x^2}\, d_x y - x\sqrt{a^2 - y^2} + \int_x \sqrt{a^2 - y^2} + C = 0,$$

or, since the part affected by the sign $\int_x$, viz.

$$\sqrt{a^2 - y^2} + \sqrt{a^2 - x^2}\, d_x y,$$

is equal to zero by the proposed,

$$y\sqrt{a^2 - x^2} + x\sqrt{a^2 - y^2} = C.$$

Ex. 2. $$1 + y + y^2 + (1 + x + x^2)\, d_x y = 0$$

$$C(x + y + 1) = 2xy + x + y - 1.$$

10. The following are instances of the integration of exact differential equations by the method of Art. 7.

Ex. 1. $$ax + by + c + (bx + my + n)\, d_x y = 0.$$

$$\therefore \quad d_{(y)} M = b = d_{(x)} N,$$

$$d_{(x)} u = ax + by + c;$$

$$\therefore\quad u = \frac{ax^2}{2} + (by + c)x + Y,$$

$$d_{(y)}u = bx + d_y Y = bx + my + n;$$

$$\therefore\quad d_y Y = my + n;$$

$$\therefore\quad Y = \tfrac{1}{2}my^2 + ny + C;$$

$$\therefore\quad \tfrac{1}{2}(ax^2 + my^2) + (by + c)x + ny + C = 0.$$

Ex. 2. $$\frac{1}{\sqrt{x^2 + y^2}} + \left(\frac{1}{y} - \frac{x}{y\sqrt{x^2 + y^2}}\right) d_x y = 0,$$

$$\log\left(x + \sqrt{x^2 + y^2}\right) + C = 0.$$

Ex. 3. $$\frac{2y - 2x d_x y}{y\sqrt{x^2 - y^2}} = 0,\quad \log\left(\frac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}}\right) + C = 0.$$

Ex. 4. $$\frac{y^2 + ax + (1 - xy)d_x y}{y^3 + 3ayx - a + a^2x^3} = 0.$$

Homogeneous Equations.

11. We come next to the case of inexact equations, in which the variables are separable by substitution; of these the most important class is homogeneous equations.

Let $M + Nd_x y = 0$ be a homogeneous equation, that is, one in which each of the functions M and N is or can be expressed by series of the form

$$M = ay^m x^{r-m} + by^n x^{r-n} + cy^p x^{r-p} + \&c.,$$

$$N = \alpha y^\mu x^{r-\mu} + \beta y^\nu x^{r-\nu} + \gamma y^\pi x^{r-\pi} + \&c.,$$

the sum of the dimensions of x and y in each term of M and N being equal to r.

Let $y = xz$ where z denotes a new function of x, then

$$M = x^r \left\{ a \left(\frac{y}{x}\right)^m + b \left(\frac{y}{x}\right)^n + \&c. \right\} = x^r f(z),$$

$$N = x^r \left\{ \alpha \left(\frac{y}{x}\right)^\mu + \beta \left(\frac{y}{x}\right)^\nu + \&c. \right\} = x^r \phi(z),$$

$$d_x y = z + x d_x z.$$

Hence, making these substitutions in the given equation, and dividing by x^r,

$$f(z) + \phi(z) \{z + x d_x z\} = 0,$$

$$\text{or} \quad \frac{1}{x} + \frac{d_x z}{\dfrac{f(z)}{\phi(z)} + z} = 0,$$

in which the variables are separated. Similarly, the variables may be separated by making $x = yz$; and the latter substitution will be more convenient when N is a more complicated expression than M.

Hence it is easy to effect the separation of the variables in equations which are either homogeneous, or can be made homogeneous; besides these, the number of equations in which that separation is possible, is very limited.

Ex. 1. $3y^2 x + 2x^3 + y^3 d_x y = 0.$

$$\text{Here} \quad \frac{M}{N} = \frac{3y^2 x + 2x^3}{y^3} = 3\frac{x}{y} + 2\left(\frac{x}{y}\right)^3 = \frac{3}{z} + \frac{2}{z^3};$$

$$\therefore \quad \frac{1}{x} + \frac{d_x z}{\dfrac{3}{z} + \dfrac{2}{z^3} + z} = 0,$$

$$\text{or} \quad \frac{1}{x} + \frac{z^3 d_x z}{z^4 + 3z^2 + 2} = 0,$$

$$\text{or}\quad \frac{1}{x} + \left(\frac{2z}{z^2+2} - \frac{z}{z^2+1}\right) d_x z = 0\,;$$

$$\therefore\quad \log x + \log(z^2+2) - \tfrac{1}{2}\log(z^2+1) = \log C\,;$$

$$\therefore\quad \frac{x(z^2+2)}{\sqrt{z^2+1}} = C\,;$$

$$\therefore\quad \frac{x\left(\dfrac{y^2}{x^2}+2\right)}{\sqrt{\dfrac{y^2}{x^2}+1}} = C,$$

$$\text{or}\quad y^2 + 2x^2 = C\sqrt{x^2+y^2}.$$

2. $y^2 + (xy + x^2)\, d_x y = 0,\qquad y = C\sqrt{1 + 2\dfrac{y}{x}}.$

3. $(x - y)\, d_x y = x + y,\qquad \tan^{-1}\dfrac{y}{x} = \log\sqrt{\dfrac{x^2+y^2}{C^2}}.$

4. $(x^2 - y^2)\, d_x y - 2xy = 0,\quad x^2 + y^2 = Cy.$

5. $x d_x y - y = \sqrt{x^2 - y^2},\qquad \sin^{-1}\dfrac{y}{x} = \log\dfrac{x}{C}.$

6. $x d_y x + y = \sqrt{x^2 + y^2},\qquad 2Cy + C^2 = x^2.$

7. $\sqrt{y} + (\sqrt{y} - \sqrt{x})\, d_x y = 0,$

$$\log(y - \sqrt{xy} + x) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\sqrt{x} - \sqrt{y}}{\sqrt{3y}}\right) = C.$$

12. In the following instances, the equations are not homogeneous, but are made so by easy substitutions.

Ex. 1. $ax + by + c + (mx + ny + p)\, d_x y = 0.$

Let $ax + by + c = z,\qquad mx + ny + p = v,$

z and v denoting functions of x; then

$$a + b d_x y = d_x z,\qquad m + n d_x y = d_x v\,;$$

2

$$\therefore \quad m + nd_z y = d_z v \,.\, d_x z = (a + bd_x y)\, d_z v\,;$$

$$\therefore \quad \frac{m - ad_z v}{bd_z v - n} = d_x y = -\frac{z}{v},$$

$$\text{or} \quad mv - nz + (bz - av)\, d_z v = 0\,;$$

this being homogeneous, assume $v = zw$, w being a function of z,

$$\therefore \quad \frac{1}{z} + \frac{(aw - b)\, d_z w}{n - (m + b)\, w + aw^2} = 0,$$

in which the variables are separated.

Ex. 2. $\quad x^m (ay + bxd_x y) = y^n (\alpha y + \beta x d_x y),$

$$\text{or} \quad (bx^m - \beta y^n)\, \frac{d_x y}{y} = (\alpha y^n - a x^m)\, \frac{1}{x}\,.$$

Let $x^m = z$, $y^n = v$, z and v being functions of x,

$$\therefore \quad \frac{m}{x} = \frac{d_x z}{z}, \quad \frac{nd_x y}{y} = \frac{d_x v}{v}, \qquad \therefore \quad \frac{nx}{my}\, d_x y = \frac{z}{v}\, d_z v\,;$$

$$\therefore \quad (bz - \beta v)\, \frac{mz}{nv}\, d_z v = \alpha v - az,$$

which is homogeneous.

Ex. 3. $\quad d_x y + ay^n x^p + by^q x^m = 0$ will become homogeneous by making $y = z^{\frac{p+1}{1-n}}$, the equation of condition between m, n, p, q, being $(p + 1)\,(1 - q) = (m + 1)\,(1 - n)$.

Ex. 4. $\quad d_x y = \dfrac{xy^3}{a^2 + xy}\,.$

This may be written

$$-\,d_x \left(\frac{1}{y}\right) = \frac{x}{\dfrac{a^2}{y} + x},$$

and therefore becomes homogeneous when z is written for $\dfrac{1}{y}$.

Linear Equations of the First Order.

13. The next important class of inexact equations of the first order which admit of being integrated, are linear equations, the general form of which is

$$d_x y + Py = Q,$$

P and Q being functions of x; they are called linear because they involve no power of y above the first.

Assume $y = vz$, v and z being functions of x,

$$\therefore \quad v d_x z + z d_x v + Pvz = Q.$$

Now z being an indeterminate quantity, may be assumed so that the equation last written down may resolve itself into two others, each of which admits of the separation of its variables; to this end let

$$v d_x z + Pzv = 0,$$

or, dividing by v, $d_x z + Pz = 0$, or $\dfrac{d_x z}{z} + P = 0$;

$$\therefore \quad \log z = -\textstyle\int_x P, \quad \text{or} \quad z = e^{-\int_x P}.$$

The remaining part of the equation gives

$$z d_x v = Q, \quad \text{or, substituting for } z, \quad d_x v = Qe^{\int_x P};$$

$$\therefore \quad v = \textstyle\int_x Qe^{\int_x P} + C,$$

$$\text{and} \quad y = e^{-\int_x P}\{\textstyle\int_x Qe^{\int_x P} + C\},$$

the complete primitive involving one arbitrary constant.

Obs. It is unnecessary to add a constant after performing the integration indicated in the equation $z = e^{-\int_x P}$; for let $z = e^{-\int_x P + C'} = C_{,}e^{-\int_x P}$; then

$$y = zv = C_{,}e^{-\int_x P}\left\{\frac{1}{C_{,}}\int_x Qe^{\int_x P} + C\right\} = e^{-\int_x P}(\textstyle\int_x Qe^{\int_x P} + CC_{,}),$$

which is the same result as before, since $CC_{,}$ is equivalent only to a single constant.

14. If we differentiate the result

$$ye^{\int_x P} = \int_x Qe^{\int_x P} + C,$$

$$\text{we get} \quad e^{\int_x P}(d_x y + Py) = e^{\int_x P} Q,$$

which shews that if we multiply the proposed equation by $e^{\int_x P}$, each member is separately integrable; and this is the most convenient practical mode of integrating it. When it is once known that the factor which makes the equation integrable is a function of x only, its value may be immediately found; for let it be denoted by X; then

$$X d_x y + (Py - Q) X = 0 \text{ is exact,}$$

$$\therefore \; d_x X = d_{(y)}(Py - Q) X = PX, \quad \text{or} \quad \frac{d_x X}{X} = P,$$

$$\therefore \; \log X = \int_x P, \quad \text{or} \quad X = e^{\int_x P}.$$

15. It must be observed that if the second member of the equation $d_x y + Py = Q$ be multiplied by any power of y, it is still reducible to the standard form of a linear equation of the first order. For suppose the equation to be

$$d_x y + Py = Qy^n,$$

then dividing both sides by y^n, and multiplying by $-(n-1)$, we get

$$d_x\left(\frac{1}{y^{n-1}}\right) - \frac{1}{y^{n-1}}(n-1) P = -Q(n-1).$$

Hence the factor which makes both sides integrable is $e^{-(n-1)\int_x P}$, and the result is

$$\frac{1}{y^{n-1}} = e^{(n-1)\int_x P}\left\{-(n-1)\int_x Qe^{-(n-1)\int_x P} + C\right\}.$$

Ex. 1. $\quad d_x y - \dfrac{x}{1+x^2}y = \dfrac{a}{1+x^2},$

$$P = -\frac{x}{1+x^2}, \quad \int_x P = -\tfrac{1}{2}\log(1+x^2) = \log\frac{1}{\sqrt{1+x^2}},$$

$$\therefore \quad e^{\int_x P} = \frac{1}{\sqrt{1+x^2}},$$

$$\therefore \quad \frac{y}{\sqrt{1+x^2}} = \int_x \frac{a}{(1+x^2)^{\frac{3}{2}}} + C = \frac{ax}{\sqrt{1+x^2}} + C;$$

$$\therefore \quad y = ax + C\sqrt{1+x^2}.$$

Ex. 2. $d_x y + y = xy^3$,

$$\text{or,} \quad d_x\left(\frac{1}{y^2}\right) - \frac{2}{y^2} = -2x,$$

$$P = -2, \quad \int_x P = -2x, \quad e^{\int_x P} = e^{-2x};$$

$$\therefore \quad \frac{e^{-2x}}{y^2} = \int_x e^{-2x}(-2x) = e^{-2x}x - \int_x e^{-2x} = e^{-2x}x + \tfrac{1}{2}e^{-2x} + C;$$

$$\therefore \quad \frac{1}{y^2} = x + \tfrac{1}{2} + Ce^{2x}.$$

3. $d_x y + \dfrac{xy}{1-x^2} = x\sqrt{y}$, $\quad \sqrt{y} = C(1-x^2)^{\frac{1}{4}} - \frac{1}{3}(1-x^2)$.

4. $d_x y + \dfrac{1-2x}{x^2} y = 1$, $\quad y = x^2(1 + Ce^{\frac{1}{x}})$.

5. $d_x y = x^3y^3 - xy$, $\quad \dfrac{1}{y^2} = x^2 + 1 + Ce^{x^2}$.

6. $d_x y = a\sin x + by$, $\quad y = -a \cdot \dfrac{b\sin x + \cos x}{1+b^2} + Ce^{bx}$.

Riccati's Equation.

16. There are certain cases of the equation

$$d_x y + by^2 = ax^m$$

(called Riccati's Equation, after the Mathematician who first considered it) in which the variables are separable.

First, let $m=0$, then $d_x y = a - by^2$, or $\dfrac{d_x y}{a-by^2} = 1$, where the variables are separated.

Secondly, let $m = -2$, and assume $y = \dfrac{u}{x}$, u being a function of x;

$$\therefore\ \frac{1}{x} d_x u - \frac{u}{x^2} + \frac{bu^2}{x^2} = \frac{a}{x^2};$$

$$\therefore\ x d_x u = a + u - bu^2,$$

where the variables are separated.

Thirdly, let $m = -4$, and assume $y = \dfrac{1}{bx} + \dfrac{u}{x^2}$,

$$\text{then} - \frac{1}{bx^2} - \frac{2u}{x^3} + \frac{d_x u}{x^2} + \frac{1}{bx^2} + \frac{2u}{x^3} + \frac{bu^2}{x^4} = \frac{a}{x^4};$$

$$\therefore\ x^2 d_x u + bu^2 = a,$$

where the variables are separated.

17. Besides the above, the variables are likewise separable in the cases when $m = \dfrac{-4i}{2i \pm 1}$, i being any integer from 0 to infinity; all which values of m evidently lie between 0 and -4.

First, let $m = \dfrac{-4i}{2i-1}$,

Assume $y = \dfrac{1}{bx} + \dfrac{1}{x^2 u}$;

$$\therefore\ d_x y = -\frac{1}{bx^2} - \frac{2}{x^3 u} - \frac{d_x u}{x^2 u^2},$$

$$by^2 - ax^m = \frac{1}{bx^2} + \frac{2}{x^3 u} + \frac{b}{x^4 u^2} - ax^m;$$

$\therefore$ adding these together,

$$0 = \frac{b}{x^4 u^2} - ax^m - \frac{d_x u}{x^2 u^2},$$

$$\text{or } x^2 d_x u + a u^2 x^{m+4} - b = 0.$$

Now, let $x = z^{\frac{1}{m+3}}$, then $d_z u = d_x u \, d_z x = (m+3) \, z^{\frac{m+2}{m+3}} \, d_z u$;

$$\therefore \ (m+3) \, z^{\frac{m+4}{m+3}} \, d_z u + a u^2 z^{\frac{m+4}{m+3}} - b = 0;$$

$$\therefore \ d_z u + \frac{a}{m+3} u^2 = \frac{b}{m+3} z^{-\frac{m+4}{m+3}}, \text{ or } d_z u + b_1 u^2 = a_1 z^{m_1}.$$

$$\text{But } m = \frac{-4i}{2i-1}, \quad \therefore \ -\frac{m+4}{m+3} = -\frac{4i-4}{2i-3} = -\frac{4(i-1)}{2(i-1)-1} = m_1.$$

Hence by these substitutions the equation is transformed into another of exactly the same form, with $i-1$ instead of i in the index of the variable in the second member.

Similarly, by substituting $\frac{1}{b_1 z} + \frac{1}{u_1 z^2}$ for u, and $z_1^{\frac{1}{m_1+3}}$ for z, we shall transform the equation into another of the same form where $m_2 = \frac{-4(i-2)}{2(i-2)-1}$; and consequently, after i substitutions the index of the variable in the second member will become zero, and the variables will be separated.

Secondly, let $m = \frac{-4i}{2i+1}$.

Assume $y = \frac{1}{u}$,

$$\text{then } -\frac{d_x u}{u^2} + \frac{b}{u^2} = a x^m,$$

$$\text{or } -d_x u + b = a x^m u^2,$$

$$\text{let } x = z^{\frac{1}{m+1}}, \text{ then } d_z u = d_x u\, d_z x = (m+1)\, z^{\frac{m}{m+1}}\, d_x u;$$

$$\therefore\ -(m+1)\, z^{\frac{m}{m+1}}\, d_z u + b = a z^{\frac{m}{m+1}}\, u^2,$$

$$\text{or } d_z u + \frac{a}{m+1} u^2 = \frac{b}{m+1} z^{-\frac{m}{m+1}},$$

$$\text{but } m = \frac{-4i}{2i+1}; \quad \therefore\ m+1 = \frac{-2i+1}{2i+1};$$

$$\therefore\ -\frac{m}{m+1} = -\frac{-4i}{-2i+1} = \frac{-4i}{2i-1}.$$

Hence by these substitutions this case is reduced to the former; and therefore when $m = \dfrac{-4i}{2i \pm 1}$, the variables in the equation $d_x y + by^2 = ax^m$ can be separated. It may be observed that the more general equation, $d_x y + by^2 x^{q-1} = ax^p$, is reducible to this form by putting $x^q = z$.

18. We shall now give some other instances of equations in which the variables are separable by particular substitutions.

Ex. 1. $\quad a(xd_x y - y) = (x + yd_x y)\sqrt{x^2 + y^2 - a^2}.$

When an equation contains the expressions

$$yd_x y + x, \quad xd_x y - y, \quad \sqrt{x^2 + y^2},$$

the introduction of polar co-ordinates will sometimes effect the separation of the variables; that is, to assume

$$x = \rho \cos\theta, \quad y = \rho \sin\theta,$$

ρ being supposed a function of θ, for then

$$yd_\theta y + xd_\theta x = \rho d_\theta \rho, \quad xd_\theta y - yd_\theta x = \rho^2.$$

Hence the proposed equation, which considering x and y as functions of θ, may be written

$$a\,(x d_\theta y - y d_\theta x) = (x d_\theta x + y d_\theta y)\sqrt{x^2 + y^2 - a^2},$$

$$\text{becomes}\quad a\rho^2 = \rho\, d_\theta \rho \sqrt{\rho^2 - a^2},$$

$$\text{or}\quad a = \frac{d_\theta \rho}{\rho}\sqrt{\rho^2 - a^2}\,;$$

$$\therefore\ a\theta = \sqrt{\rho^2 - a^2} - a \sec^{-1}\frac{\rho}{a} + C,$$

$$\text{or}\quad a\tan^{-1}\frac{y}{x} = \sqrt{x^2 + y^2 - a^2} - a\sec^{-1}\frac{\sqrt{x^2+y^2}}{a} + C.$$

Ex. 2. $$y - x d_x y + d_x y \sqrt{(ax + by)\frac{x}{y}} = 0.$$

$$\text{or}\quad y d_y x - x + \sqrt{(ax + by)\frac{x}{y}} = 0\,;$$

$$\text{let}\quad \frac{x}{y} = z,\quad z \text{ being a function of } y,$$

$$\text{then}\quad y d_y x - x = y^2 d_y z\,;$$

$$\therefore\ y^2 d_y z + \sqrt{(az + b)\,zy} = 0,$$

where the variables are separated.

Ex. 3. $$d_x y = \frac{a^2 + x^2 - y^2}{a^2}.$$

Since this is satisfied by $y = x$, assume $y = x + z^{-1}$,

$$\therefore\ 1 - z^{-2} d_x z = 1 + \frac{1}{a^2}\left(-2xz^{-1} - z^{-2}\right),$$

or $d_x z - \dfrac{2xz}{a^2} = \dfrac{1}{a^2}$, which is a linear equation;

3

$$\therefore \quad z e^{-\frac{x^2}{a^2}} = \frac{1}{a^2} \int_x e^{-\frac{x^2}{a^2}} + C,$$

$$\text{or} \quad \frac{e^{-\frac{x^2}{a^2}}}{y - x} = \frac{1}{a^2} \int_x e^{-\frac{x^2}{a^2}} + C.$$

Ex. 4. $(1 - xy)\, d_x y + y^2 + ax = 0.$

$$\text{Assume} \quad y = \frac{z - ax^2}{1 + xz}, \quad \text{so that} \quad z = \frac{y + ax^2}{1 - xy};$$

$$\text{then} \quad \frac{d_z z}{z^3 - a} + \frac{x}{1 + ax^3} = 0,$$

where the variables are separated.

Ex. 5. $(y - x)\, d_x y = \dfrac{n(1 + y^2)^{\frac{3}{2}}}{\sqrt{1 + x^2}}.$

$$\text{assume} \quad y = \frac{x - z}{1 + xz};$$

$$\therefore \quad \frac{z\, d_z z}{(1 + z^2)(z + n\sqrt{1 + z^2})} - \frac{1}{1 + x^2} = 0.$$

Euler's Equation.

19. To integrate the equation

$$d_x y \sqrt{a + bx + cx^2 + ex^3 + fx^4} + \sqrt{a + by + cy^2 + ey^3 + fy^4} = 0;$$

or, considering x and y as functions of a new variable t,

$$d_t y \sqrt{X} + d_t x \sqrt{Y} = 0.$$

Let the function of t which expresses x be determined by the equation $d_t x = \sqrt{X}$, and therefore that which expresses y by the equation $d_t y = -\sqrt{Y}$; also let $x + y = p$, $x - y = q$, p and q being functions of t.

Then since $(d_t x)^2 = X$; $\therefore\ 2d_t x\, d_t^2 x = d_t X$, or $d_t^2 x = \frac{1}{2} d_x X$; similarly $d_t^2 y = \frac{1}{2} d_y Y$.

$$\therefore\ d_t^2 p = \tfrac{1}{2}(d_x X + d_y Y)$$

$$= \tfrac{1}{2}\left\{2b + 2c(x+y) + 3e(x^2+y^2) + 4f(x^3+y^3)\right\}$$

$$= b + c(x+y) + \frac{3e}{4}\left\{(x+y)^2 + (x-y)^2\right\}$$

$$+ 2f(x+y)\frac{(x+y)^2 + 3(x-y)^2}{4}$$

$$= b + cp + \frac{3e}{4}(p^2+q^2) + \tfrac{1}{2}fp(p^2+3q^2),$$

and $d_t p \,.\, d_t q = X - Y = b(x-y) + c(x^2-y^2) + e(x^3-y^3) + f(x^4-y^4)$

$$= bq + cpq + \frac{e}{4}q(3p^2+q^2) + \tfrac{1}{2}fpq(p^2+q^2);$$

$$\therefore\ d_t^2 p - \frac{1}{q} d_t p \,.\, d_t q = \frac{e}{2}q^2 + fpq^2,$$

$$\text{or}\quad d_t\left\{\left(\frac{d_t p}{q}\right)^2\right\} = e d_t p + 2fp d_t p;$$

$$\therefore\ \frac{(d_t p)^2}{q^2} = C + ep + fp^2;$$

$$\therefore\ d_t p = q\sqrt{C + ep + fp^2},$$

$$\text{or}\quad \sqrt{X} - \sqrt{Y} = (x-y)\sqrt{C + e(x+y) + f(x+y)^2},$$

the integral required. The discovery of this integral, which is due to Euler, was of great importance, as being the first step towards the foundation of the Theory of Elliptic Functions.

20. To integrate the equation

$$\sqrt{1 - c^2\sin^2\psi} + \sqrt{1 - c^2\sin^2\phi}\; d_\phi \psi = 0,$$

or considering ϕ and ψ as functions of another variable t,

$$\sqrt{1 - c^2\sin^2\psi}\; d_t\phi + \sqrt{1 - c^2\sin^2\phi}\; d_t\psi = 0.$$

Let $d_t\phi = \sqrt{1-c^2\sin^2\phi}$, and $\therefore\ d_t\psi = -\sqrt{1-c^2\sin^2\psi}$;

$$\therefore\ (d_t\phi)^2 = 1 - c^2\sin^2\phi,\quad (d_t\psi)^2 = 1 - c^2\sin^2\psi;$$

$$\therefore\ d_t^2\phi + d_t^2\psi = -\tfrac{1}{2}c^2(\sin 2\phi + \sin 2\psi),$$

$$d_t^2\phi - d_t^2\psi = -\tfrac{1}{2}c^2(\sin 2\phi - \sin 2\psi).$$

Let $p = \phi + \psi,\quad q = \phi - \psi,$

$$\therefore\ d_t^2 p = -c^2\sin p\cos q,$$

$$d_t^2 q = -c^2\cos p\sin q,$$

and $d_t p\,.\,d_t q = (d_t\phi)^2 - (d_t\psi)^2 = -c^2(\sin^2\phi - \sin^2\psi)$

$$= \frac{c^2}{2}(\cos 2\phi - \cos 2\psi) = -c^2\sin p\sin q.$$

$$\therefore\ \frac{d_t^2 p}{d_t p} = \frac{\cos q\, d_t q}{\sin q};\quad \therefore\ \log(d_t p) = \log(C\sin q);$$

$$\therefore\ d_t p = C\sin q;\quad \text{similarly}\quad d_t q = C\sin p;$$

$$\therefore\ \sqrt{1-c^2\sin^2\phi} \mp \sqrt{1-c^2\sin^2\psi} = C\sin(\phi\mp\psi)$$

is the integral of the proposed equation; which is only Euler's equation under a different form.

The equations $\sqrt{1+y^4} + \sqrt{1+x^4}\,d_x y = 0,$

$$\sqrt{y-y^3} + \sqrt{x-x^3}\,d_x y = 0,$$

are immediately reducible to the above form, viz.

$$\sqrt{1-\tfrac{1}{2}\sin^2\psi} + \sqrt{1-\tfrac{1}{2}\sin^2\phi}\,d_\phi\psi = 0;$$

the former by making $x = \tan\frac{1}{2}\phi$, $y = \tan\frac{1}{2}\psi$; the latter by making $\sqrt{x} = \cos\phi$, $\sqrt{y} = \cos\psi$.

On the Factors which render integrable a Differential Equation of the First Order.

21. The most natural way of obtaining the complete integral of a differential equation of the first order, is to prepare it so that its first member may become an exact differential coefficient; for then we shall have only to integrate and add a constant. This preparation is always possible by means of a factor, when the equation is reduced to the form $d_x y + K = 0$. For let an equation $f(x, y, C) = 0$ be resolved with respect to C, so that

$$C = \phi(x, y);$$

$$\therefore \text{ by differentiation, } 0 = P + Qd_x y, \text{ or } d_x y + \frac{P}{Q} = 0.$$

Now the equation $M + Nd_x y = 0$ may be put under the form $d_x y + K = 0$, which agrees with the preceding, and may consequently be supposed to have arisen from the elimination of a constant between the primitive $f(x, y, C) = 0$, and its immediately derived equation. On this supposition, therefore,

$d_x y + K = 0$ is identical with $d_x y + \dfrac{P}{Q} = 0$;

$$\therefore d_x y + K = \frac{P + Qd_x y}{Q},$$

$$\text{or } d_x \phi(x, y) = Q(d_x y + K), \text{ identically.}$$

The second member therefore is an exact differential coefficient, which proves that there always exists a factor proper to render the expression $d_x y + K$ integrable.

22. But although the existence of the factor in every case is thus established, the investigation of it is usually attended with greater difficulties than the solution of the original equation.

For let $P + Qd_x y = 0$ be an exact differential equation; and let z, a function of x and y, be a common factor of P and Q, so that $P = Mz$, $Q = Nz$, by the removal of which, the equation is reduced to the inexact state,

$$M + Nd_x y = 0;$$

then because $P + Qd_xy = 0$ is exact,

$$d_{(y)}P = d_{(x)}Q, \quad \text{or} \quad d_{(y)}(Mz) = d_{(x)}(Nz),$$

$$\text{or} \quad zd_{(y)}M + Md_{(y)}z = zd_{(x)}N + Nd_{(x)}z,$$

$$\text{or} \quad Nd_{(x)}z - Md_{(y)}z = z(d_{(y)}M - d_{(x)}N);$$

an equation between x, y, z, and the partial differential coefficients of z, for determining the factor z. The consideration of this equation in its general state must be reserved till we come to treat of partial differential equations of the first order; but the following particular cases may be noticed.

23. First, suppose that the factor is a function of only one of the variables x, then $d_{(y)}z = 0$, and the equation becomes

$$\frac{d_xz}{z} = \frac{1}{N}(d_{(y)}M - d_{(x)}N),$$

which, being integrated, gives z; for the hypothesis requires that the second member should be independent of y.

Similarly, if the factor be a function of y only, it will result from the integration of

$$\frac{d_yz}{z} = \frac{1}{M}(d_{(x)}N - d_{(y)}M),$$

of which the second member is independent of x.

Hence, if in any equation $M + Nd_xy = 0$ we find

$$\frac{1}{N}(d_{(y)}M - d_{(x)}N) = X$$

a function of x only, or

$$\frac{1}{M}(d_{(x)}N - d_{(y)}M) = Y$$

a function of y only; the factors which may make it integrable are respectively $e^{\int_x X}$, $e^{\int_y Y}$.

Ex. 1. $d_x y + (Py - Q) = 0$, the linear equation of the first order.

This compared with $M + Nd_x y = 0$, gives

$$M = Py - Q, \quad N = 1;$$

$$\therefore\ d_{(y)}M - d_{(x)}N = P, \text{ and } \frac{1}{N}(d_{(y)}M - d_{(x)}N) = P,$$

a function of x only; therefore the factor is $e^{\int_x P}$.

Ex. 2.
$$y^2 + (1 - xy)\, d_x y = 0,$$

$$M = y^2, \qquad N = 1 - xy,$$

$$d_{(x)}N - d_{(y)}M = -3y;$$

$$\therefore\ \int_y \frac{1}{M}(d_{(x)}N - d_{(y)}M) = -\int_y \frac{3}{y} = \log\frac{1}{y^3},$$

therefore the factor is $\dfrac{1}{y^3}$.

24. In the case of homogeneous equations, a factor proper to render them integrable, is readily discovered by means of the property that if u be a homogeneous function of n dimensions of the independent quantities t and z, then

$$nu = td_t u + zd_z u.$$

For suppose V, a homogeneous function of x and y of m dimensions, to be a factor which makes $M + Nd_x y$ an exact differential coefficient, M and N being homogeneous functions of x and y of r dimensions; then if U denote the primitive, it will be homogeneous and of $m + r + 1$ dimensions, and we shall have

$$VM + VNd_x y = d_x U, \quad d_{(x)}U = VM, \quad d_{(y)}U = VN,$$

$$xVM + yVN = (m + r + 1)\, U;$$

$$\therefore\ \frac{M + Nd_x y}{Mx + Ny} = \frac{1}{m + r + 1} \cdot \frac{d_x U}{U},$$

and as the second member is an exact differential coefficient, it follows that the first is so likewise, and consequently, that $M + Nd_x y$ is made exact by means of the multiplier

$$\frac{1}{Mx + Ny}.$$

25. The property of homogeneous functions assumed above is easily proved. Let u be a homogeneous function of the independent quantities t and z of n dimensions; then if we change t into $t(1 + h)$ and z into $z(1 + h)$, u will become

$$u(1 + h)^n = u + nuh + \&c.$$

But by Taylor's theorem, u will also become

$$u + d_t u \cdot ht + d_z u \cdot hz + \&c.$$

therefore, equating the coefficients of h,

$$nu = td_t u + zd_z u.$$

And, generally, if u be a homogeneous function of n dimensions of any number of independent quantities t, z, w, &c., and we change them into $t(1 + h)$, $z(1 + h)$, $w(1 + h)$, &c., the new value of u will be equally expressed by $u(1 + h)^n$ or by $e^{(td_t + zd_z + \ldots)h} u$; and equating the coefficients of h^r in these two identical expressions, we get, separating as above the symbols of operation from those of quantity,

$$n(n-1)\ldots(n-r+1)\, u = (td_t + zd_z + wd_w + \ldots)^r u.$$

Ex. $xy + y^2 + (xy - x^2)\, d_x y = 0.$

The factor is

$$\frac{1}{(xy + y^2)\, x + (xy - x^2)\, y} = \frac{1}{2y^2 x};$$

$$\therefore \frac{xy + y^2}{2y^2 x} + \frac{xy - x^2}{2y^2 x}\, d_x y = 0,$$

is an exact differential coefficient, and gives the primitive by Art. 7.

$$\frac{x}{2y} + \tfrac{1}{2}\log(xy) + C = 0.$$

26. Whenever the variables can be separated in an equation, a factor which makes it integrable can also be found.

For suppose that $M + Nd_xy = 0$, by the introduction of two other variables u and z, is transformed into $R + Sd_zu = 0$, so that

$$M + Nd_xy = R + Sd_zu;$$

and suppose V to be a function of u and z, such that if we divide $R + Sd_zu$ by it, the variables are separated, i. e. $\frac{R}{V}$ contains z only, and $\frac{S}{V}$ contains u only;

$$\therefore \quad \frac{1}{V}(M + Nd_xy) = \frac{R}{V} + \frac{S}{V} d_zu$$

is an exact differential coefficient; and consequently $\frac{1}{V}$, which, upon restoring the values of u and z, becomes a function of x and y, is a factor which makes $M + Nd_xy = 0$ integrable.

Ex. 1. $a + bx^2y^2 + x^2d_xy = 0.$

Assuming $y = \frac{u}{x}$, we find

$$a + bx^2y^2 + x^2d_xy = a + bu^2 + xd_xu - u,$$

and dividing by $x(a - u + bu^2)$ we get

$$\frac{a + bx^2y^2 + x^2d_xy}{x(a - u + bu^2)} = \frac{1}{x} + \frac{d_xu}{a - u + bu^2};$$

$$\therefore \quad \frac{1}{x(a - u + bu^2)} = \frac{1}{ax - x^2y + bx^3y^2}$$

is a factor which makes the proposed equation integrable.

Ex. 2. $y^2 + ax + (1 - xy)\, d_xy = 0.$

By assuming $y = \dfrac{z - ax^2}{1 + xz}$, it may be shewn that a factor which makes the proposed integrable is

$$\frac{1}{y^3 + 3ayx - a + a^2x^3}.$$

Ex. 3. $M + N d_x y = 0$, a homogeneous equation.

In this case we know, that making $y = xz$, we have

$$M = x^r f(z), \quad N = x^r \phi(z),$$

$$\text{and} \quad M + N d_x y = x^r f(z) + x^r \phi(z) \{z + x d_x z\};$$

consequently, dividing by $x^{r+1} \{f(z) + z\phi(z)\} = Mx + Ny$, we get

$$\frac{M + N d_x y}{Mx + Ny} = \frac{1}{x} + \frac{\phi(z)\, d_x z}{f(z) + z\phi(z)};$$

$$\therefore \quad \frac{1}{Mx + Ny}$$

is a factor which makes the proposed equation integrable.

Obs. That $\dfrac{M + N d_x y}{Mx + Ny}$ is an exact differential coefficient, provided M and N be homogeneous functions of x and y of the same dimensions, admits of an easy proof as follows.

We must shew that $d_{(y)}\left(\dfrac{M}{Mx + Ny}\right) = d_{(x)}\left(\dfrac{N}{Mx + Ny}\right)$.

Now putting $\dfrac{y}{x} = z$, we have

$$\frac{M}{Mx + Ny} = \frac{1}{x} \frac{1}{1 + \dfrac{Ny}{Mx}} = \frac{1}{x} \cdot \frac{1}{1 + F(z)}$$

$$\frac{N}{Mx + Ny} = \frac{1}{y}\left(1 - \frac{Mx}{Mx + Ny}\right) = \frac{1}{y} - \frac{1}{y} \frac{1}{1 + F(z)};$$

$$\therefore \quad d_{(y)}\left(\frac{M}{Mx+Ny}\right) = -\frac{1}{x} \cdot \frac{d_z F(z) \cdot \frac{1}{x}}{\{1+F(z)\}^2},$$

$$d_{(x)}\left(\frac{N}{Mx+Ny}\right) = -\frac{1}{y} \cdot \frac{d_z F(z) \cdot \frac{y}{x^2}}{\{1+F(z)\}^2},$$

which expressions are evidently equal to one another.

27. There always exists an infinite number of factors which render an equation of the first order and degree integrable.

For let z be the factor by means of which the equation

$$M + N d_x y = 0$$

is made integrable, and $u = 0$ its complete primitive, so that

$$z(M + N d_x y) = d_x u;$$

multiply both sides by $F(u)$, where $F(u)$ denotes any function of u, and we have

$$z F(u)(M + N d_x y) = F(u) d_x u;$$

and as the second member is an exact differential coefficient, it follows that the first is so likewise; therefore $z F(u)$ is a factor which makes the proposed equation integrable, whatever form be assigned to $F(u)$.

28. The following geometrical problems are added to illustrate this part of the subject.

I. To determine the trajectory of a given family of curves.

Let AA', BB', (fig. 1.) be two of a family of curves resulting from the equation $f(X, Y, c) = 0$, by given particular values to the constant c; and let AB be a curve which

cuts AA', BB', and all the curves resulting from the equation by giving all possible values to c, at the same angle; then AB is called the trajectory of this family of curves. Let x, y, be the co-ordinates of the point A in AB, between which we are required to find a relation; AT', AT, tangents to the curve and trajectory at A, $\tan TAT' = a$; and let a value $\psi(X, Y)$ of $d_X Y$ be obtained from the equation $f(X, Y, c) = 0$, not involving c; then at the point A, $\tan AT'N = \psi(x, y)$, and $\tan ATN = d_x y$,

$$\therefore \; a = \frac{d_x y - \psi(x, y)}{1 + d_x y \cdot \psi(x, y)},$$

the differential equation to the curve AB; and as it does not involve c, AB will cut every curve in the series at an angle whose tangent $= a$. The equation when integrated will involve an arbitrary constant, and consequently will represent a system of curves, every one of which cuts the former system at the same angle; the constant may be determined, if a point through which the trajectory is to pass, be given. If the angle TAT' be a right angle, or (a) infinite, the differential equation to the trajectory, which is then called orthogonal, is

$$1 + d_x y \cdot \psi(x, y) = 0.$$

Ex. 1. To find the orthogonal trajectory to all curves resulting from the equation $y^2(c - x) = x^3$, by giving all possible values to c.

$$\text{Here } c - x = \frac{x^3}{y^2}, \qquad \therefore \; -1 = \frac{3x^2}{y^2} - \frac{2x^3}{y^3}\, d_x y;$$

$$\therefore \; d_x y = \psi(x, y) = \frac{y^3 + 3x^2 y}{2x^3},$$

therefore, substituting for $\psi(x, y)$ its value, the differential equation to the trajectory is

$$2x^3 + (y^3 + 3x^2 y)\, d_x y = 0;$$

a homogeneous equation whose integral is

$$x^2 + y^2 = C\sqrt{2x^2 + y^2}.$$

Similarly, let $f(\rho, \theta, c) = 0$ be the equation to the curve AA' referred to polar co-ordinates, and let it give for $\rho d_\rho \theta$ the value $\psi(\rho, \theta)$ independent of c; then considering ρ and θ as co-ordinates of the point A in the curve AB,

$$\tan SAT' = \psi(\rho, \theta), \qquad \tan SAT = \rho d_\rho \theta,$$

$$\therefore \quad a = \frac{\psi(\rho, \theta) - \rho d_\rho \theta}{1 + \psi(\rho, \theta)\rho d_\rho \theta},$$

which is the differential equation to the trajectory; or if it be orthogonal,

$$1 + \psi(\rho, \theta)\rho d_\rho \theta = 0.$$

Ex. 2. Let the curves be a system of circles touching a straight line in the same point, then taking that point as the origin and measuring θ from the line, their equation is

$$\rho = c \sin\theta,$$

$$\therefore \frac{1}{\rho} = \frac{\cos\theta}{\sin\theta} d_\rho \theta, \quad \text{or} \quad \rho d_\rho \theta = \frac{\sin\theta}{\cos\theta} = \psi(\rho, \theta);$$

$$\therefore 1 + \frac{\sin\theta}{\cos\theta}\rho d_\rho \theta = 0, \quad \text{or} \quad \cos\theta + \sin\theta \rho d_\rho \theta = 0,$$

$$\text{or} \quad d_\rho \left(\frac{\rho}{\cos\theta}\right) = 0; \qquad \therefore \rho = C\cos\theta,$$

the equation to the orthogonal trajectory, which represents a system of circles passing through the given point and having the given line for their diameter.

We may generalize this problem, by finding the orthogonal trajectory of all circles described through two given points.

II. To determine a curve such, that the locus of the extremity of its polar subtangent shall be a straight line.

The polar subtangent is a line drawn from the origin perpendicular to the radius vector to meet the tangent.

Let ρ, θ, be the polar co-ordinates of any point P in the curve sought (fig. 2); then those of the extremity T of

its polar subtangent will be $\rho^2 d_\rho \theta$ and $\theta - \frac{\pi}{2}$, which must satisfy the equation to a straight line, viz.

$$\rho' = c \sec (\theta' - a);$$

$$\therefore\ \rho^2 d_\rho \theta = c \sec \left(\theta - a - \frac{\pi}{2}\right) = \frac{c}{\sin (\theta - a)}.$$

$$\therefore\ \frac{c}{\rho^2} = \sin (\theta - a)\, d_\rho \theta, \quad \text{and } \frac{c}{\rho} = \cos (\theta - a) + C;$$

$$\therefore\ \rho = \frac{c}{C + \cos (\theta - a)},$$

the equation to curves of the second order, having the pole for one of their foci.

III. To find a curve in which SG varies as SP, PG being a normal at P, and SG a fixed line through S. (fig. 2).

Taking SG for the axis of x, the equation to the normal at P is

$$(Y - y)\, d_x y + X - x = 0;$$

therefore making

$$Y = 0, \quad X = SG = x + y d_x y,$$

$$\text{and } SP = \sqrt{x^2 + y^2}, \quad \therefore\ x + y d_x y = e\sqrt{x^2 + y^2},$$

$$\text{or } \sqrt{x^2 + y^2} = ex + C,$$

the equation to a curve of the second order.

IV. To find a curve which is always cut by its radius vector at an angle proportional to the corresponding angle of revolution; that is, $\angle\, SPT \propto \angle\, ASP$, (fig. 2).

Let ρ, θ, be the co-ordinates of any point in the curve, then the angle at which the radius vector cuts the curve, has for its tangent $\rho d_\rho \theta$;

$$\therefore\ \rho d_\rho \theta = \tan n\theta, \quad \text{or } \frac{d_\theta \rho}{\rho} = \frac{\cos n\theta}{\sin n\theta}, \quad \therefore\ \left(\frac{\rho}{C}\right)^n = \sin n\theta.$$

SECTION II.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER, BUT NOT OF THE FIRST DEGREE.

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29. WHEN a differential equation of the first order is of a higher degree than the first, we know that it is not obtained by the direct differentiation of its primitive, but results from eliminating a constant, (which enters into the primitive in a dimension above the first,) between the primitive and its derived equation; the degree of the differential equation and the dimension of the constant eliminated above the lowest dimension in which it appears, being always the same. The general form of such equations free from radicals, is

$$(d_x y)^n + p_1 (d_x y)^{n-1} + p_2 (d_x y)^{n-2} + \dots + p_{n-1} d_x y + p_n = 0,$$

the coefficients being functions of x and y.

If this can be resolved with respect to $d_x y$ into its (n) simple factors, it will assume the form

$$(d_x y + q_1)\,(d_x y + q_2) \dots (d_x y + q_n) = 0;$$

then each of these factors put equal to zero, will be an equation of the first order and degree, whose integral may be found by the methods of the preceding section; and any one of these integrals, as well as the continued product of any number of them, will evidently satisfy the proposed equation. If, therefore, we integrate the n equations,

$$d_x y + q_1 = 0, \quad d_x y + q_2 = 0, \dots \quad d_x y + q_n = 0,$$

and complete them all with the same constant C, as the proposed equation is of the first order, we shall obtain the required primitive involving C in the n^{th} power, by equating their continued product to zero.

Ex. 1. $$(d_x y)^2 + \frac{2x}{y} d_x y - 1 = 0;$$

$$\therefore \quad d_x y + \frac{x}{y} = \pm \sqrt{1 + \frac{x^2}{y^2}},$$

$$\text{or} \quad \frac{y\,d_x y + x}{\sqrt{y^2 + x^2}} = \pm 1;$$

$$\therefore \quad + \sqrt{y^2 + x^2} = x + C, \text{ and } - \sqrt{y^2 + x^2} = x + C;$$

$$\therefore \quad (\sqrt{y^2 + x^2} - C - x)(\sqrt{y^2 + x^2} + C + x) = 0,$$

$$\text{or} \quad y^2 + x^2 - (C + x)^2 = 0,$$

$$\text{or} \quad y^2 = 2Cx + C^2.$$

Ex. 2. $$x^2 (d_x y)^2 - 2xy\,d_x y + y^2 - x^2 y^2 - x^4 = 0.$$

$$\therefore \quad x d_x y - y = \pm x \sqrt{x^2 + y^2},$$

$$\text{or} \quad \frac{d_x \left(\frac{y}{x}\right)}{\pm \sqrt{1 + \left(\frac{y}{x}\right)^2}} = 1.$$

$$\therefore \quad \log \left\{ \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} \right\} = x + C,$$

$$\text{and} \quad -\log \left\{ \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} \right\} = \log \left\{ \sqrt{1 + \left(\frac{y}{x}\right)^2} - \frac{y}{x} \right\} = x + C;$$

$$\therefore \quad \sqrt{x^2 + y^2} + y = cxe^x, \text{ changing the constant,}$$

$$\text{and} \quad \sqrt{x^2 + y^2} - y = cxe^x;$$

$$\therefore \quad (\sqrt{x^2 + y^2} - cxe^x + y)(\sqrt{x^2 + y^2} - cxe^x - y) = 0,$$

$$\text{or} \quad y = \frac{x}{2} \left(\frac{1}{ce^x} - ce^x \right).$$

30. When the resolution of the proposed equation into its simple factors is impossible, there are still various forms for which the complete primitive can be determined, or its determination made to depend on elimination; this is done by means of substitution, or differentiation, or other analytical artifices, of which we shall now give some instances.

Obs. The arbitrary constant in what follows is often reserved under the sign of integration.

31. When the equation contains only one of the variables, x suppose, and can be solved with respect to that variable, so that $x = f(d_x y)$; let $d_x y$ be denoted by p, then $x = f(p)$; and integrating the equation $d_x y = p$ by parts, we get

$$y = xp - \int_x x d_x p = pf(p) - \int_p f(p);$$

between which and the equation $x = f(p)$, eliminating p, we shall obtain the required integral.

Similarly, if we have $y = f(p)$, since

$$d_p x = d_y x \, . \, d_p y = \frac{1}{p} d_p f(p),$$

we shall have to eliminate p between $y = f(p)$,

$$\text{and } x = \int_p \frac{1}{p} d_p f(p).$$

Ex. 1. $x + x(d_x y)^2 = 1$,

$$x = \frac{1}{1+p^2},$$

$$y = \frac{p}{1+p^2} - \tan^{-1} p + C;$$

$$\therefore \quad y = \sqrt{x(1-x)} - \tan^{-1}\sqrt{\frac{1-x}{x}} + C.$$

Ex. 2. $y = a\sqrt{1+p^2}, \quad x + C = a \log(\sqrt{y^2 - a^2} + y)$.

This is the solution of the problem in which it is required to find a curve such that the perpendicular on the tangent from the foot of the ordinate shall be constant.

Ex. 3. $y\sqrt{1+p^2} = ap$,

$$d_p y = \frac{a}{(1+p^2)^{\frac{3}{2}}};$$

$$\therefore \quad x + C = \int_p \frac{a}{p\,(1+p^2)^{\frac{3}{2}}} = \frac{a}{\sqrt{1+p^2}} + a\log\left(\frac{p}{\sqrt{1+p^2}+1}\right),$$

$$\text{or} \quad x + C = \sqrt{a^2 - y^2} + a\log\left(\frac{y}{\sqrt{a^2-y^2}+a}\right).$$

This is the solution of the problem to find a curve such that the tangent terminated at the axis of x shall be of a constant length.

32. An equation not coming immediately under this case, may sometimes be reduced to it by putting $p = xz$, or $p = yz$.

Ex. $(d_x y)^3 + ax\,d_x y + x^3 = 0$. Let $p = xz$,

$$\text{then} \quad x(z^3+1) + az = 0, \quad \text{or} \quad x = -\frac{az}{1+z^3};$$

$$\therefore \quad d_z y = d_x y \,.\, d_z x = -\frac{a^2 z^2 (2z^3 - 1)}{(z^3+1)^3},$$

and z must be eliminated between the integral of this, and the equation $x = -\dfrac{az}{1+z^3}$.

33. When the equation contains both the variables x and y, provided it be homogeneous with respect to them, we may assume $\dfrac{y}{x} = z$; then the equation will take the form, (which is not solvable with respect to p by supposition,)

$$f(z, p) = 0.$$

Suppose this capable of being solved with respect to z, and let it give $z = \phi(p)$; now $y = xz$ gives $p = z + x\,d_x z$,

$$\text{or} \quad \frac{1}{x} = \frac{d_p z}{p - z};$$

substitute the above value of z, and integrate this equation; then p must be eliminated between the result, which will be of the form $\log x = F(p)$, and $y = x\,\phi(p)$.

Ex. $y - xd_xy = nx\sqrt{1+(d_xy)^2}$;

$$\therefore\quad \frac{y}{x} = p + n\sqrt{1+p^2};$$

$$\therefore\quad \frac{1}{x} = \frac{d_x(p+n\sqrt{1+p^2})}{-n\sqrt{1+p^2}} = -\frac{1}{n}\left(\frac{d_xp}{\sqrt{1+p^2}} + n\frac{d_x\sqrt{1+p^2}}{\sqrt{1+p^2}}\right);$$

$$\therefore\quad \log x = -\frac{1}{n}\{\log(p+\sqrt{1+p^2}) + n\log\sqrt{1+p^2}\} + \log C;$$

$$\therefore\quad x = \frac{C(p+\sqrt{1+p^2})^{-\frac{1}{n}}}{\sqrt{1+p^2}},\quad \frac{y}{x} = p + n\sqrt{1+p^2},$$

between which equations p must be eliminated.

This is the solution of the problem, to find a curve such that the perpendicular upon the tangent from the origin shall vary as the abscissa to the point of contact.

Let $n = 1$. $\frac{C}{x} = 1 + p^2 + p\sqrt{1+p^2},\quad \frac{y}{x} = p + \sqrt{1+p^2};$

$$\therefore\quad \frac{y^2 + (C-x)^2}{x^2} = (1+p^2)(p+\sqrt{1+p^2})^2 = \frac{C^2}{x^2},$$

$$\text{or}\quad y^2 = 2Cx - x^2.$$

34. Another integrable form is $y = xd_xy + f(d_xy)$, which is called Clairaut's form, after the Mathematician who first considered it. Substituting p for d_xy, and differentiating, we get

$$y = xp + f(p),$$

$$d_xy = p = p + xd_xp + d_pf(p)\,.\,d_xp;$$

$$\therefore\quad \{x + d_pf(p)\}\,d_xp = 0,$$

which resolves itself into the two

$$x + d_pf(p) = 0,\quad d_xp = 0.$$

The first of these gives $p = \phi(x)$ suppose; this value substituted for p in the proposed equation, furnishes a relation between x and y which satisfies the proposed equation, but which involves no arbitrary constant, and cannot therefore be the complete primitive. The other equation must therefore lead to the complete primitive; but this gives $p = C$, and by substituting this value of p in the proposed we find

$$y = Cx + f(C).$$

Hence Clairaut's form has the property, that the complete primitive is obtained by substituting the arbitrary constant C for p, in that form. If we integrate $p=C$, we find $y=Cx+C'$; but the condition of the proposed equation being satisfied gives $C' = f(C)$, the same result as before.

We shall afterwards return to the consideration of the other solution, which is called the singular solution, and is not derivable from the complete integral.

Ex. 1. $y = xp + \dfrac{a(1+p^2)}{p}$.

Differentiating, we get $p = p + xd_xp + a\left(1 - \dfrac{1}{p^2}\right)d_xp$,

$$\text{or} \quad \left(x + a - \frac{a}{p^2}\right)d_xp = 0;$$

$\therefore \quad d_xp = 0$ gives $p = C$, and $y = Cx + \dfrac{a(1+C^2)}{C}$,

the complete integral, and

$$x + a - \frac{a}{p^2} = 0 \text{ gives } p = \pm\sqrt{\frac{a}{x+a}},$$

which, substituted in $py = a + p^2(x+a)$, gives

$$\pm y\sqrt{\frac{a}{x+a}} = 2a, \text{ or } y^2 = 4a(x+a),$$

the singular solution.

Ex. 2. $y = xp + \sqrt{b^2 + a^2p^2}$,

$$y = Cx + \sqrt{b^2 + C^2a^2}, \text{ the complete integral.}$$

$$a^2y^2 + b^2x^2 = a^2b^2 \text{ the singular solution.}$$

This is the solution of the problem to find a curve, such that the product of the perpendiculars dropped from two given points upon the tangent may be invariable; for taking the line joining the two given points (whose distance suppose $= 2c$) for the axis of x, and their middle point for origin, and x, y the co-ordinates of any point in the curve, the equation to the tangent at that point will be

$$Y - y = d_x y\,(X - x), \quad \text{or} \quad Y = d_x y X + (y - x d_x y),$$

and the lengths of the perpendiculars dropped upon this line from the points (c, o), $(-c, o)$ will be

$$\frac{-pc - (y - xp)}{\sqrt{1 + p^2}}, \quad \frac{pc - (y - xp)}{\sqrt{1 + p^2}};$$

and the product of these is

$$\frac{(y - xp)^2 - c^2p^2}{1 + p^2} = b^2, \text{ suppose};$$

$$\therefore \quad y = xp + \sqrt{b^2 + a^2p^2}, \text{ putting } a^2 = b^2 + c^2.$$

35. A still more general case is the equation

$$y = xf(p) + \phi(p),$$

which by differentiation is reduced to a linear equation of the first order in x; for we get

$$p = f(p) + x d_p f(p)\, d_x p + d_p \phi(p)\, d_x p;$$

$$\therefore \quad \{p - f(p)\}\, d_p x = x d_p f(p) + d_p \phi(p);$$

$$\therefore \quad d_p x + x \frac{d_p \phi(p)}{f(p) - p} = -\frac{d_p \phi(p)}{f(p) - p},$$

which gives $x = F(p)$; then p must be eliminated between this and the proposed equation.

Ex. 1. $y = xp^2 + 2p$,

$$p = p^2 + 2xp\,d_x p + 2\,d_x p,$$

$$\text{or} \quad d_p x + \frac{2x}{p-1} = -\frac{2}{p^2 - p},$$

which is made integrable by the factor $(p-1)^2$;

$$\therefore \quad x(p-1)^2 = -\int_p \frac{2p-2}{p} = -2p + \log p^2 + C.$$

But $p = -\frac{1}{x} \pm \sqrt{\frac{y}{x} + \frac{1}{x^2}}$; therefore substituting this in the preceding, we obtain the complete primitive between x and y.

Ex. 2. $y = xmp + n\sqrt[3]{1 + p^3}$;

$$xp^{\frac{m}{m-1}} = -\frac{n}{m-1}\int_p \frac{p^{\frac{2m-1}{m-1}}}{(1+p^3)^{\frac{2}{3}}}.$$

Ex. 3. $y + p(a - x) = n\int_x \sqrt{1 + p^2}$;

$$y = \frac{(a-x)^{n+1}}{2c^n(n+1)} + \frac{c^n}{2(n-1)(a-x)^{n-1}} + C'.$$

This is the solution of the problem of finding the path of a point P which moves uniformly towards another point Q, also moving uniformly in a straight line.

For taking A (fig. 3.) for the origin, and AB, which is perpendicular to By the line in which Q moves, for the axis of x, we have, supposing P and Q to start together from A and B, $BQ = nAP$, or if $AN = x$, $NP = y$, $AB = a$,

$$y + (a - x)p = n\int_x \sqrt{1 + p^2}.$$

36. In the following examples the method of substitution succeeds.

Ex. 1. $(1-p^2)\,xy = p\,(x^2 - y^2 - c^2)$,

which expresses that the normal bisects the angle between the focal distances; $2c$ being the distance of the foci, the origin at the middle point between them, and the line joining them the axis of x.

$$\text{Let } p = \frac{xz}{y}, \qquad \therefore\ y^2 = x^2 z - c^2 \frac{z}{1+z};$$

therefore, differentiating,

$$\left\{x^2 - \frac{c^2}{(1+z)^2}\right\} d_x z = 0;$$

this resolves itself into

$$x = \pm \frac{c}{1+z}, \text{ which gives } y^2 + (x-c)^2 = 0,$$

the singular solution;

and $d_x z = 0$, or $z = C$, which gives $y^2 = C\left(x^2 - \dfrac{c^2}{1+C}\right)$,

the complete integral.

If we integrate $p = \dfrac{Cx}{y}$, we get $y^2 = Cx^2 + C'$, where C' must be determined by the condition of the proposed equation being satisfied; and by this condition, in general whenever the method of solution raises the order of the equation, must the number of constants be reduced.

By the same substitution may be solved the more general form

$$axyp^2 + p\,(x^2 - ay^2 - b) - xy = 0.$$

Ex. 2. $x d_x y - y = X \sqrt{(d_x y)^2 - \dfrac{2y}{x} d_x y + 1}$,

$$\text{or } d_x y - \frac{y}{x} = \frac{X}{x} \sqrt{\left(d_x y - \frac{y}{x}\right)^2 + 1 - \frac{y^2}{x^2}}.$$

Let $y = xz$, then $d_x y = z + x d_x z = \frac{y}{x} + x d_x z$;

$$\therefore\ x^2 (d_x z)^2 = \frac{X^2}{x^2} \{x^2 (d_x z)^2 + 1 - z^2\};$$

$$\therefore\ \frac{d_x z}{\sqrt{1 - z^2}} = \frac{X}{x\sqrt{x^2 - X^2}}.$$

If $X = 1$, we have $\sin^{-1} z = \sec^{-1} x + C$,

$$\text{or}\ \sin^{-1} \frac{y}{x} = \sec^{-1} x + C.$$

Ex. 3. and 4.

$$\frac{y - xp}{\sqrt{1 + p^2}} = f(\sqrt{x^2 + y^2}),$$

$$\frac{y - xp}{\sqrt{x^2 + y^2}\ \sqrt{1 + p^2}} = f\left(\frac{x}{\sqrt{x^2 + y^2}}\right).$$

Introducing polar co-ordinates, we get for the first,

$$\frac{\rho^2}{\sqrt{(d_\theta \rho)^2 + \rho^2}} = f(\rho), \qquad \text{or}\ d_\rho \theta = \frac{f(\rho)}{\rho\sqrt{\rho^2 - \{f(\rho)\}^2}}.$$

The second gives

$$\frac{\rho}{\sqrt{(d_\theta \rho)^2 + \rho^2}} = f(\cos\theta), \qquad \text{or}\ \frac{d_\theta \rho}{\rho} = \frac{\sqrt{1 - \{f(\cos\theta)\}^2}}{f(\cos\theta)}.$$

The former expresses that the perpendicular on the tangent from the origin is a given function of the radius vector; the latter, that the sine of the angle at which the radius vector cuts the curve is a given function of the cosine of the angle at which it is inclined to the axis of x.

SECTION III.

ON THE SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS.

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37. From the complete integral of a differential equation we can deduce as many particular integrals as we please, by giving to the arbitrary constant particular values. But some differential equations are satisfied by a relation between x and y not containing an arbitrary constant, and not deducible from the complete integral; such a relation is called, as has been said, a singular solution of the differential equation. The existence of such solutions depends upon the fact, that when a solution of a differential equation has been obtained in any manner, it will still be a solution after a quantity of any kind has been introduced in any way, provided the same derived equation result. This is merely an extension of the principle on which the arbitrary constant is added.

38. Before entering upon the general theory, it may be useful to consider the following particular instance.

Let the equation

$$y = xd_xy + a + a(d_xy)^2 \quad (1)$$

be proposed, which, since it falls under Clairaut's form, has for its complete integral

$$y = Cx + a + aC^2; \quad (2)$$

C being the arbitrary constant. If we now regard C, not as a constant, but as a function of x, and differentiate, we get

$$d_xy = C + (x + 2aC)\, d_xC;$$

and if we eliminate C between this and $y = Cx + a + aC^2$, we shall obtain a differential equation, but not the proposed one,

for that arises by eliminating C by means of the equation $d_x y = C$. But if C be so determined as to make the coefficient of $d_x C$ vanish, that is, if $C = -\frac{x}{2a}$, then the derived equation will be $d_x y = C$, and the result of the elimination of C will be the proposed equation.

Substituting for C its value, we get

$$y = -\frac{x^2}{2a} + a + \frac{x^2}{4a} = -\frac{x^2}{4a} + a,$$

a relation which manifestly satisfies the proposed equation; for it gives $d_x y = -\frac{x}{2a}$, and these values of y and $d_x y$, being substituted in the proposed equation, make it identical. But this solution contains no arbitrary constant, and yet being the equation to a parabola it cannot, either by making $C = 0$, or any other constant quantity, arise from the complete integral which is the equation to a straight line; it is consequently a singular solution, and arises from the complete integral by changing C into a function of x so determined as to make the term involving $d_x C$ disappear from the value of $d_x y$.

Thus we see how the singular solution arises from the complete integral; next, let us consider its geometrical signification. The proposed differential equation expresses that the curves to which it belongs have the property, that the tangent at any point is intersected by a perpendicular upon it from a given point in a given straight line.

Take the given point S (fig. 4.) for the origin, and TS, AS respectively parallel and perpendicular to the given line AC, for the axes of x and y. Let TC be the tangent at a point whose co-ordinates are x and y, then its equation is

$$Y - y = d_x y\,(X - x),$$

and the equation to the perpendicular upon it from S is

$$Y d_x y = -X,$$

and for their point of intersection

$$Y\{1 + (d_x y)^2\} - y = - x d_x y,$$

but this point is always in AC for which $Y = a$;

$$\therefore\ y - x d_x y = a + a(d_x y)^2,$$

the same as the proposed equation.

The complete integral $y = Cx + a(1 + C^2)$, which represents a series of straight lines, evidently satisfies the problem for all values of C; for let Tt be any one of these lines, then $y = -\dfrac{x}{C}$ is the equation to a line through S perpendicular to it; and combining the equations to get the co-ordinates of their point of intersection, we have

$$y = -C^2 y + a(1 + C^2), \quad \text{or} \quad y = a;$$

the intersection consequently falls in AC. And as a straight line is its own tangent at every point, the equation

$$y = Cx + a(1 + C^2), \text{ for all values of } C$$

represents a line such that the intersection of the tangent at any point, and a perpendicular upon it from S, falls in the given line AC. Now the curve which is generated by the perpetual intersections of these lines will also satisfy the problem; for each of the lines will be a tangent to it, and therefore perpendiculars from S upon its tangents will intersect them in the various points of AC. To get the equation to this curve we must, according to the usual method, differentiate with respect to the parameter C, which gives $0 = x + 2Ca$, and eliminate C between this, and the equation $y = Cx + a(1 + C^2)$, which gives

$$y = -\frac{x^2}{4a} + a, \quad \text{or} \quad 4a(a - y) = x^2,$$

the equation to a parabola, vertex A, focus S, of which curve it is a well known property, that the perpendicular

from the focus intersects the tangent at any point, in the tangent at the vertex.

This result being obtained by exactly the same process as the singular solution was obtained, of course coincides with it; hence it appears that the singular solution belongs to the curve which touches the family of curves resulting from the complete integral by making the arbitrary constant assume all possible values. The conclusions arrived at in this particular instance, we shall now shew to hold generally.

39. Having given the complete integral of a differential equation, to find its singular solution.

Let $f(x, y, d_x y) = 0$ be a proposed differential equation, and suppose it to result from the elimination of the arbitrary constant c, between the equation $F(x, y, c) = 0$, and its immediately derived equation $M + N d_x y = 0$. Now change c into c', any function of x and y, then our equation becomes $F(x, y, c') = 0$; and its immediately derived equation

$$M' + N' d_x y + C' d_x c' = 0 \;\; (1),$$

c' entering into M' and N' just as c did into M and N, and $d_x c'$ of course denoting $d_{(x)} c' + d_{(y)} c' \cdot d_x y$; now C' is the differential coefficient of $F(x, y, c')$ with respect to c', regarding x and y as constant, and will therefore usually involve x, y, and c'; and if put equal to zero, will give such a value for c' as makes the last term of equation (1) disappear; and then the elimination of c' must evidently produce the proposed equation $f(x, y, d_x y) = 0$. Let this value of c' be substituted in $F(x, y, c') = 0$; then this equation is changed into $\phi(x, y) = 0$, and furnishes a relation between x and y which satisfies the equation $f(x, y, d_x y) = 0$, but contains no arbitrary constant, and is not deducible from the complete integral by giving a particular value to the constant; since it results from the complete integral by substituting for c a variable value deduced from the equation $d_c F(x, y, c) = 0$. Consequently, the relation $\phi(x, y) = 0$ is the singular solution required.

40. To explain the geometrical signification of the singular solution of a differential equation.

Let $F(x, y, c) = 0$ (1) be the complete integral of a differential equation between two variables; if we differentiate it with regard to c, we have $d_c F(x, y, c) = 0$ (2); and if between these equations we eliminate c, we get $\phi(x, y) = 0$ (3), where c does not appear, and which is a singular solution of the differential equation, of which (1) is the complete primitive. Suppose equation (1) to be the equation to a system of curves, in which the position and dimensions of any particular curve is defined by a particular value of the parameter c; also let equation (3) be the equation to a curve referred to the same co-ordinate axes. Now equations (1) and (2), when c receives a certain value, are satisfied by the same values of x and y. Hence from the manner of its formation, equation (3) is satisfied by the same values; or the curves which are represented by (3) and (1), with a particular value of c, have a common point. But equations (3) and (1), being each a solution of the same differential equation, furnish the same value of $d_x y$ for the same values of x and y; consequently the curves touch one another at their common point. The same thing happens for every one of the system of curves which equation (1) represents. Therefore the curve represented by the singular solution touches in a point every curve represented by the complete primitive. This is the geometrical interpretation of the singular solution of a differential equation of the first order.

41. Having given a solution of a differential equation, to find whether it is included in the complete integral or not.

Let $d_x y = f(x, y)$ be the proposed differential equation, and $y = F(x, c)$ its complete integral, c being the arbitrary constant; and when $c = c'$, let this become $y = u$, u containing no arbitrary constant; then $y = u$ is a particular integral of the proposed.

Hence, since $F(x, c) - u$ becomes zero when $c = c'$, we have

$$F(x, c) - u = (c - c')^m . z = az, \text{ suppose,}$$

z being a function of x and c, which is neither infinite nor zero, when $c=c'$, or when $a=0$; and m expressing the highest power of $c-c'$ which enters into every term of $F(x, c)-u$. Consequently the complete integral becomes

$$y=u+az,$$

which being substituted in the proposed equation,

$$d_xy=f(x, y),$$

$$\text{gives} \quad d_xu+ad_xz=f(x, u+az) \quad (1).$$

Now since z is neither infinite nor zero when $a=0$, we may expand it in a series of ascending powers of a, in the form

$$z=K+Aa^{\alpha}+Ba^{\beta}+\text{\&c.},$$

α, β, &c. being increasing and positive, and K, A, B, &c. functions of x;

$$\therefore \quad d_xu+ad_xz=d_xu+d_xKa+d_xAa^{\alpha+1}+\text{\&c.}$$

Again, the development of $f(x, u+az)$ will be of the form

$$f(x, u+az)=f(x, u)+M(az)^m+N(az)^n+\text{\&c.},$$

m, n, &c. being increasing and positive. Hence, by substitution in equation (1), observing that $d_xu=f(x, u)$, we get

$$d_xK\,.\,a+d_xA\,.\,a^{\alpha+1}+\text{\&c.}$$

$$=Ma^m(K+Aa^{\alpha}+\text{\&c.})^m+Na^n(K+Aa^{\alpha}+\text{\&c.})^n+\text{\&c.}$$

Now unless this equation is identical, $y=u$ cannot result from $y=f(x, c)$ by changing c into c'; and $\therefore$ $y=u$ cannot be a particular integral of the proposed; and if it satisfies the proposed, it must be a singular solution. Now the indices m, n, &c., are known, for they result from writing $u+az$ for y in $f(x, y)$, and expanding according to power of az; and we must endeavour to determine α, β, &c. so that the

two members of the equation may be identical. If m be > 1, this can easily be effected; for we must have $d_x K = 0$, or $K =$ a constant; $\alpha + 1 = m$, and $d_x A = MK^m$; and so on for the other terms. Consequently it will be possible to make the two members identical, and $y = u$ will be a particular integral. In the same way the identity may be established if $m = 1$. But if $m < 1$, there is no term on the first side corresponding to $MK^m \alpha^m$; and since K cannot be equal to zero, it is impossible to satisfy the identity; and therefore $y = u$ is a singular solution. Hence to discover whether a given solution, $y = u$, of a differential equation $d_x y = f(x, y)$, is a singular solution or not; we must write $u + h$ for y in the value of $d_x y$, and if the expansion in ascending powers of h involve a power of h, whose index is < 1, the solution in question is a singular solution.

42. To deduce the singular solutions from the differential equation, without knowing its complete primitive.

Let $y = u$ be a singular solution of the equation

$$d_x y = f(x, y);$$

then by the preceding article, substituting $u + h$ for y, we get

$$f(x, u + h) = f(x, u) + Mh^m + Nh^n + \&c.$$

where m, n, &c. are proper fractions;

$$\therefore\ d_u f(x, u + h) = d_h f(x, u + h) = mMh^{m-1} + nNh^{n-1} + \&c.;$$

consequently, when $h = 0$, $d_u f(x, u) = \infty$.

But $d_u f(x, u)$ is what $d_y f(x, y)$ becomes when $y = u$; and therefore, conversely, every value, u of y, which satisfies $d_x y = f(x, y)$, and makes $d_y f(x, y) = \infty$, is a singular solution of the equation $d_x y = f(x, y)$.

43. It is not essential to give the equation the explicit form $d_x y = f(x, y)$. For let $d_x y = p$, and let $V = 0$ be the given relation between x, y and p; then we may regard p as a function of x and y determined by the equation $V = 0$;

$$\therefore \quad d_{(y)}V + d_{(p)}V d_y p = 0, \quad \text{or} \quad d_y p = -\frac{d_{(y)}V}{d_{(p)}V}.$$

Hence the condition $d_y f(x, y) = \infty$ is equivalent to $d_{(p)}V = 0$, provided $d_{(y)}V$ remains finite; and therefore the singular solutions of the equation $V = F(x, y, p) = 0$, are determined by eliminating p between $V = 0$ and $d_{(p)}V = 0$, provided always that these solutions do not make $d_{(y)}V = 0$. It is evident that if we consider y as the independent variable, and put the equation under the form

$$V = F(x, y, d_y x) = 0,$$

the same reasonings would shew that singular solutions may be obtained by eliminating $p' = d_y x$, between $V = 0$ and $d_{p'}V = 0$.

Ex. 1. To find the singular solution of

$$y - x d_x y + x - \frac{y}{d_x y} = a.$$

Here $d_{(p)}V = -x + \dfrac{y}{p^2} = 0$; $\therefore$ $p^2 = \dfrac{y}{x}$, and the proposed becomes

$$(x + y - a) p = y + xp^2, \quad \text{or} \quad (x + y - a) p = 2y,$$

$$\text{or} \quad (x + y - a)^2 = 4xy,$$

which may be reduced to the form $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Ex. 2. To find the singular solution of

$$(y - x d_x y)\left(x - \frac{y}{d_x y}\right) = a^2. \quad 4xy = a^2.$$

Ex. 3. To find the singular solution of

$$(y - x d_x y)^2 + \left(x - \frac{y}{d_x y}\right)^2 = a^2. \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

These three examples determine respectively the curves which have the properties that $OT+OT'$ is constant, that area of triangle TOT' is constant, and that TT' is constant, TT' being the tangent at any point meeting the axes Ox, Oy in T and T'. (fig. 5.)

Ex. 4. $y+\{x-\tan^{-1}(d_x y)\}\, d_x y - 1 = 0.$

$$\text{Here}\quad d_{(p)}V = x - \tan^{-1}p - \frac{p}{1+p^2} = 0;$$

$$\therefore\quad x = \tan^{-1}p + \frac{p}{1+p^2};$$

$$y = \frac{1}{1+p^2},\quad \therefore\quad x = \cos^{-1}\sqrt{y} + \sqrt{y-y^2},$$

which represents a cycloid whose base coincides with the axis of x, the origin being in the centre of the base. This is the solution of the problem to find a curve always touched by the same diameter of a circle rolling along a straight line.

Ex. 5. $d_x y = \dfrac{x}{\sqrt{x^2+y^2-a^2}-y};$

$$\therefore\quad d_y p = \frac{-x}{(\sqrt{x^2+y^2-a^2}-y)^2}\left(\frac{y}{\sqrt{x^2+y^2-a^2}}-1\right);$$

$\therefore$ the relation $x^2+y^2-a^2=0$ makes $d_y p$ infinite, and satisfies the proposed equation; it is consequently a singular solution of the proposed.

Ex. 6. To find whether or not $x^2+y^2-a^2=0$ is a singular solution of $d_x y = \dfrac{x}{\sqrt{x^2+y^2-a^2}-y}$. The solution gives $y=\sqrt{a^2-x^2}$, therefore substituting $\sqrt{a^2-x^2}+h$ for y in the value of $d_x y$, we get

$$\frac{x}{\sqrt{2h\sqrt{a^2-x^2}+h^2}-\sqrt{a^2-x^2}-h} = \frac{-x}{\sqrt{a^2-x^2}}\left\{1+\frac{\sqrt{2h}}{(a^2-x^2)^{\frac{1}{4}}}-\&c.\right\},$$

when developed according to powers of h; and as the index of h is a proper fraction, $x^2+y^2-a^2=0$ is a singular solution.

44. Every factor proper to make a proposed differential equation integrable, is made infinite by the singular solution.

Let $d_x y + f(x, y) = 0$ be the proposed equation, $F(x, y) = c$ its complete integral, and z the factor which makes it integrable, so that

$$z\left\{d_x y + f(x, y)\right\} = d_x F(x, y);$$

also, let $y = u$ be the singular solution; then this is not deducible from the complete integral, and therefore if u be written for y in $F(x, y)$, the result will not be constant; if therefore we substitute u for y, in the preceding equation, since the second member will have a finite value, and the factor $d_x y + f(x, y)$ of the first member will be zero, the value of z corresponding to this substitution must be infinite.

This property will sometimes lead to the discovery of the factor which makes an equation integrable; as in the example

$$(a^2 - x^2)\, d_x y + xy = a\sqrt{x^2 + y^2 - a^2},$$

a singular solution of which is $x^2 + y^2 - a^2 = 0$; if we try a factor of the form

$$(x^2 - a^2)^m\, (y^2 + x^2 - a^2)^n,$$

we arrive at $m = -1$, $n = -\frac{1}{2}$; and the factor which makes the proposed integrable is

$$(x^2 - a^2)^{-1}\, (x^2 + y^2 - a^2)^{-\frac{1}{2}}.$$

SECTION IV.

DIFFERENTIAL EQUATIONS OF THE SECOND ORDER, AND OF HIGHER ORDERS.

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45. EVERY differential equation of the n^{th} order admits of a primitive with n arbitrary constants.

Let $f(x, y, c_1, c_2, \dots c_n) = 0$ be an equation between the variables x and y, containing n constants $c_1, c_2, \dots c_n$. Let the first n derived equations be

$$f_1(x, y, d_x y, c_1, \dots c_n) = 0,$$

$$f_2(x, y, d_x y, d_x^2 y, c_1, \dots c_n) = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f_n(x, y, d_x y, d_x^2 y, \dots d_x^n y, c_1, \dots c_n) = 0.$$

Between these n equations and the original, the n constants may be eliminated, and the result will be

$$F(x, y, d_x y, d_x^2 y, \dots d_x^n y) = 0, \quad (1)$$

a differential equation in which none of the constants enter.

Conversely, a differential equation of the n^{th} order being proposed, it must admit of a primitive containing n arbitrary constants, because this number of constants, and no more, can be eliminated in its formation. Hence every differential equation of the n^{th} order admits of a primitive containing n arbitrary constants.

46. Again, between the original equation and its first $n - 1$ derived equations, $n - 1$ of the constants may be eliminated, and a differential equation of the $(n - 1)^{\text{th}}$ order with one constant will result.

Every such differential equation, having the same primitive with equation (1), is a first integral of that equation; hence a differential equation of the n^{th} order has n first integrals, each a differential equation of the $(n-1)^{th}$ order, and containing one constant.

Also, between the original equation and the first (r) of its derived equations, (r) of the constants may be eliminated, and a differential equation of the r^{th} order containing $n-r$ constants, will result, which is an integral of equation (1). Now (r) constants can be eliminated in a number of ways equal to the number of combinations of n things taken r together, or

$$\frac{n(n-1)\dots(n-r+1)}{1.2.3\dots r}.$$

This then is the number of integrals of equation (1) of the $(n-r)^{th}$ order, each a differential equation of the r^{th} order, and containing $n-r$ constants.

47. Of the general equation of the second order

$$F(d_x^2 y,\ d_x y,\ y,\ x) = 0,$$

we shall first of all consider the following particular cases, in which $d_x^2 y$ is involved with only one, or two, of the other quantities x, y, $d_x y$; and which admit of integration, or rather of reduction to forms of the first order.

I. $F(d_x^2 y,\ x) = 0$. Let this by resolution, give

$$d_x^2 y = f(x), \qquad \therefore\ d_x y = \int_x f(x) + C;$$

and integrating again, and adding another constant, we obtain the complete integral. The same process applies to $d_x^n y = f(x)$. Also, if we have $F(d_x^{n+1} y,\ d_x^n y) = 0$, and put $d_x^n y = u$, we get $F(d_x u,\ u) = 0$; and if this can be integrated, and gives $u = f(x)$, it is reduced to the case just noticed.

Ex. $x^2 d_x^2 y = a, \qquad \therefore\ d_x^2 y = \dfrac{a}{x^2}, \quad d_x y = -\dfrac{a}{x} + C;$

$$\therefore\ y = a \log\frac{1}{x} + Cx + C'.$$

II. $F(d_x^2 y, d_x y) = 0$. Let this by resolution, give

$$d_x^2 y = f(d_x y), \quad \text{or} \quad d_x p = f(p), \quad \text{putting } d_x y = p;$$

$$\therefore \; d_p x = \frac{1}{f(p)}, \quad x = \int_p \frac{1}{f(p)};$$

$$\text{and } d_p y = d_x y \,.\, d_p x = \frac{p}{f(p)}, \quad \therefore \; y = \int_p \frac{p}{f(p)};$$

p must be eliminated between these two equations.

Ex. $a d_x^2 y = d_x y$; $\therefore\; d_x p = \dfrac{p}{a}$; $\therefore\; \log p = \dfrac{x}{a} + C$,

$$\text{and } d_p y = d_x y \,.\, d_p x = p \,.\, \frac{a}{p} = a; \quad \therefore \; y = ap + C';$$

$$\therefore \; \frac{x}{a} + C = \log \frac{y - C'}{a}.$$

III. $F(d_x^2 y, y) = 0$. Let this give $d_x^2 y = f(y)$;

$$\therefore \; 2 d_x y d_x^2 y = 2 f(y)\, d_x y, \quad \text{and} \quad (d_x y)^2 = C + 2\int_y f(y);$$

$$\therefore \; d_x y = \sqrt{C + 2\int_y f(y)}, \quad \text{and} \quad x = \int_y \frac{1}{\sqrt{C + 2\int_y f(y)}}.$$

Also $F(d_x^{n+2} y, d_x^n y) = 0$, putting $d_x^n y = u$, becomes

$$F(d_x^2 u, u) = 0;$$

and this, treated as above, gives $x = \phi(u)$; and if $x = \phi(u)$ can be solved with respect to u, it is brought under Case I.

Ex. 1. $a^2 d_x^2 y + y = 0$; $\therefore\; 2a^2 d_x y d_x^2 y + 2y d_x y = 0$;

$$\therefore \; a^2 (d_x y)^2 + y^2 = C; \quad \therefore \; \frac{a d_x y}{\sqrt{C - y^2}} = 1;$$

$$\therefore \; a \sin^{-1} \frac{y}{\sqrt{C}} = x + C'.$$

Ex. 2. $\sqrt{ay}\, d_x^2 y = 1.$

$$\frac{x}{\sqrt[4]{a}} = \frac{2}{3}(\sqrt{y} + C)^{\frac{3}{2}} - 2C\sqrt{\sqrt{y} + C} + C'.$$

IV. $F(d_x^2 y, d_x y, x) = 0.$

This becomes of the first order in p and x, by putting p for $d_x y$ and $d_x p$ for $d_x^2 y$; let its integral be $\phi(x, p, C) = 0$.

If this by resolution give p or $d_x y = f(x)$, then $y = \int_x f(x)$; among other cases, this will happen when the proposed is of the form $d_x^2 y + P d_x y = Q$; for it can be solved as a linear equation of the first order.

If it gives $x = f(p)$, then

$$y = \int_x p = xp - \int_x x d_x p = xp - \int_p f(p);$$

and p must be eliminated between these equations.

Ex. 1. $d_x^2 y + \frac{1}{x} d_x y = 0; \quad \therefore\ x d_x^2 y + d_x y = d_x(x d_x y) = 0;$

$$\therefore\ x d_x y = C; \quad \therefore\ y = C \log x + C'.$$

Ex. 2. $(1 + x^2) d_x^2 y + 1 + (d_x y)^2 = 0;$

$$\therefore\ \frac{d_x p}{1 + p^2} + \frac{1}{1 + x^2} = 0, \quad \tan^{-1} p + \tan^{-1} x = \tan^{-1} c;$$

$$\therefore\ p = \frac{c - x}{1 + cx} = \frac{1}{c}\left(\frac{1 + c^2}{1 + cx} - 1\right);$$

$$\therefore\ y = \frac{1 + c^2}{c^2}\log(1 + cx) - \frac{x}{c} + C'.$$

Ex. 3. $\frac{\{1 + (d_x y)^2\}^{\frac{3}{2}}}{d_x^2 y} = f(x); \quad \therefore\ \frac{d_x p}{(1 + p^2)^{\frac{3}{2}}} = \frac{1}{f(x)};$

$$\therefore \frac{p}{\sqrt{1+p^2}} = \int_x \frac{1}{f(x)} = X, \text{ suppose};$$

$$\therefore \quad p = \frac{X}{\sqrt{1-X^2}}, \quad \text{and} \quad y = \int_x \frac{X}{\sqrt{1-X^2}}.$$

This is the solution of the inverse problem of the radius of curvature, in which it is required to find the curve whose radius of curvature is any given function of the abscissa.

Ex. 4. $d_x^2 y + (e^x - 1) d_x y = e^{2x}$, $y = e^x + Ce^{-e^x} + C'$.

V. $F(d_x^2 y, d_x y, y) = 0$.

Putting $d_x y = p$, we get $d_x^2 y = d_y p \cdot d_x y = p d_y p$; and the substitution of $p d_y p$ for $d_x^2 y$, and of p for $d_x y$, will make the proposed of the first order in p and y; let its integral be

$$\phi(p, y, C) = 0.$$

If this by resolution give $p = d_x y = f(y)$, then

$$x = \int_y \frac{1}{f(y)}.$$

If it gives $y = f(p)$, then

$$x = \int_y \frac{1}{p} = \frac{y}{p} + \int_y \frac{y}{p^2} d_y p = \frac{y}{p} + \int_p \frac{f(p)}{p^2},$$

and p must be eliminated between these equations.

Ex. 1. $y d_x^2 y + (d_x y)^2 = 0$,

or $d_x(y d_x y) = 0$; $\therefore$ $y d_x y = C$; $\therefore$ $y^2 = 2Cx + C'$.

Ex. 2. $y d_x^2 y + n(d_x y)^2 + n = 0$.

$$\therefore \quad y p d_y p + n p^2 + n = 0;$$

$$\therefore \quad \tfrac{1}{2}\log(1+p^2) + n \log y = \tfrac{1}{2}\log C;$$

$$\therefore \quad (1+p^2)\,y^{2n} = C;$$

$$\therefore \quad p = \sqrt{Cy^{-2n} - 1}, \quad \text{and} \quad x = \int_y \frac{y^n}{\sqrt{C - y^{2n}}}.$$

This is the solution of the problem in which it is required to find a curve whose radius of curvature shall vary as its normal; for this condition gives

$$y\sqrt{1+p^2} = \frac{n\,(1+p^2)^{\frac{3}{2}}}{\pm\, d_x p}, \quad \text{or} \quad y d_x p + (\mp n)(1+p^2) = 0,$$

$-$ or $+$ according as the curve is convex or concave to the axis of x. If $n = 1$ the curve is a circle, if $n = 2$ a cycloid, if $n = -1$ a common catenary.

Ex. 3. $\quad 1 + (d_x y)^2 - 2y d_x^2 y = 3ay\,\{1 + (d_x y)^2\}^{\frac{3}{2}}.$

$$\text{this gives} \quad d_y x = \frac{ay^{\frac{3}{2}} + C}{\sqrt{y - (ay^{\frac{3}{2}} + C)^2}}.$$

Ex. 4. $\quad y^2 d_x^2 y + \sqrt{1 + (d_x y)^2} = 0$, gives $d_y x = \dfrac{y}{\sqrt{(Cy+1)^2 - y^2}}.$

VI. $\quad F(d_x^2 y, y, x) = 0.$

As there is no substitution by which this can be generally reduced to an equation of the first order between two variables, the artifice to be employed in any case will depend upon the nature of the example proposed. Among other substitutions for $d_x^2 y$, the two following may be noticed,

$$x d_x^2 y = d_x\left(x^2 d_x \cdot \frac{y}{x}\right), \quad \text{and} \quad x d_x^2 y = d_x^2(xy) - 2d_x y.$$

Ex. 1. $\quad x^2 d_x^2 y = 2y;$

$$\therefore \quad d^2(xy) - 2d_x y = \frac{2y}{x},$$

$$\text{or} \quad d^2(xy) = \frac{2}{x}\, d_x(xy);$$

$$\therefore \quad d_x(xy) = Cx^2;$$

$$\therefore \quad y = \frac{Cx^2}{3} + \frac{C'}{x}.$$

Ex. 2. $(x^2 + y^2)^2 d_x^2 y + a^2 y = 0,$

$$\text{or} \quad d_x\left(x^2 d_x \frac{y}{x}\right) + \frac{a^2 xy}{(x^2+y^2)^2} = 0,$$

$$\text{or} \quad 2\left(x^2 d_x \frac{y}{x}\right) d_x \left\{x^2 d_x \left(\frac{y}{x}\right)\right\} + \frac{2a^2 \frac{y}{x} d_x \frac{y}{x}}{\left\{1 + \left(\frac{y}{x}\right)^2\right\}^2} = 0;$$

$$\therefore \quad \left(x^2 d_x \frac{y}{x}\right)^2 - \frac{a^2}{1 + \left(\frac{y}{x}\right)^2} = C, \quad \text{let } \frac{y}{x} = z;$$

$$\therefore \quad x^2 d_x z = \sqrt{C + \frac{a^2}{1+z^2}},$$

$$\text{or} \quad -\frac{1}{x} = \int_z \frac{\sqrt{1+z^2}}{\sqrt{a^2 + C'(1+z^2)}}.$$

Ex. 3. $d_x^2 y = ax + by;$

$$\text{or} \quad d_x^2(ax+by) = b(ax+by),$$

which becomes $d_x^2 z = bz$, putting $ax + by = z$, and so falls under Case III.

48. When the equation

$$F(x, y, d_x y, d_x^2 y) = 0, \quad \text{or } F(x, y, p, d_x p) = 0,$$

is homogeneous, reckoning the dimensions of p and $d_x p$ to be 0 and -1 respectively; it may be reduced to an equation of the first order by putting

$$y = xz, \quad \text{and } d_x p = \frac{q}{x}.$$

For each term, if r denote its dimensions, will consist of some function of p multiplied by a factor of the form

$$\left(\frac{y}{x}\right)^m x^{n+r}(d_x p)^n,$$

m and n being any numbers from 0 to ∞; therefore upon making the substitutions stated above, every term will be divisible by x^r, and the equation will assume the form

$$F(p,\ q,\ z) = 0;$$

and if this can be solved relative to q, we shall have

$$q = \phi(z,\ p); \quad \text{but} \quad q = x d_x p = (p - z)\frac{d_z p}{d_x z} = (p - z)\, d_z p;$$

$$\therefore\ \phi(z,\ p) + (z - p)\, d_z p = 0;$$

let this give

$$p = \psi(z), \qquad \text{then} \quad \psi(z) = z + x d_x z,$$

which will give the required integral

$$z = \frac{y}{x} = f(x).$$

Ex. $\quad n\{1 + (d_x y)^2\}^{\frac{3}{2}} = d_x^2 y \sqrt{x^2 + y^2};$

putting $d_x y = p, \quad y = xz, \quad d_x p = \frac{q}{x}$, we find

$$n(1 + p^2)^{\frac{3}{2}} = q\sqrt{1 + z^2};$$

$$\therefore\ q = \frac{n(1 + p^2)^{\frac{3}{2}}}{\sqrt{1 + z^2}} = (p - z)\, d_z p,$$

which is the same as Ex. 5. Art. 18

Linear Equations of the Second and Higher Orders.

49. The linear equation of the n^{th} order is,

$$d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y + \ldots + p_{n-1} d_x y + p_n y = X,$$

all the coefficients being functions of x, and each term of the first member involving either y, or one of its differential coefficients, in the first power. The first step towards the integration of this equation is the establishment of the following theorem.

If there be n particular values u_1, u_2, $u_3 \ldots u_n$, functions of x, which, when substituted for y, satisfy the equation

$$d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y + \ldots + p_n y = 0,$$

its complete integral is

$$y = a_1 u_1 + a_2 u_2 + a_3 u_3 + \ldots + a_n u_n,$$

a_1, a_2, ... a_n being arbitrary constants.

For let this value of y be substituted in the expression

$$d_x^n y + p_1 d_x^{n-1} y + \ldots + p_n y,$$

and it becomes

$$d_x^n (a_1 u_1 + a_2 u_2 + \ldots + a_n u_n)$$

$$+ p_1 d_x^{n-1} (a_1 u_1 + a_2 u_2 + \ldots + a_n u_n) + \ldots + p_n (a_1 u_1 + a_2 u_2 + \ldots + a_n u_n),$$

or, collecting the terms multiplied by the factors a_1, a_2, ... a_n,

$$a_1 \left\{ d_x^n u_1 + p_1 d_x^{n-1} u_1 + \ldots + p_n u_1 \right\}$$

$$+ a_2 \left\{ d_x^n u_2 + p_1 d_x^{n-1} u_2 + \ldots + p_n u_2 \right\} + \ldots$$

$$+ a_n \left\{ d_x^n u_n + p_1 d_x^{n-1} u_n + \ldots p_n u_n \right\};$$

Now, since u_1, u_2, ... u_n satisfy the equation, each of the quantities within brackets is equal to zero, and therefore the whole is identically zero; and therefore the assumed value of y satisfies the equation; and it contains n arbitrary constants, consequently it is the complete integral of the equation.

Thus the equation $d_x^2 y - n^2 y = 0$ is satisfied by $y = e^{nx}$, and since it is not altered by changing the sign of n, it is also satisfied by $y = e^{-nx}$; $\therefore$ $y = a_1 e^{nx} + a_2 e^{-nx}$ is the complete integral.

Linear Equations with constant Coefficients.

50. To integrate the equation

$$d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y + \ldots + p_n y = u,$$

all the coefficients and u being constant.

First write $y + \dfrac{u}{p_n}$ instead of y, and the equation becomes

$$d_x^n y + p_1 d_x^{n-1} y + \ldots + p_n y = 0.$$

Let $y = e^{mx}$, $\therefore$ $e^{mx}(m^n + p_1 m^{n-1} + \ldots + p_n)$ is the value of the first member; now this will vanish if m be any root of the equation

$$m^n + p_1 m^{n-1} + p_2 m^{n-2} + \ldots + p_n = 0, \quad \text{or} \quad M = f(m) = 0,$$

which is called the auxiliary equation of the proposed linear equation.

Hence the n real or imaginary roots of this equation, m_1, m_2, m_3 ... m_n, provided they be all unequal, will give n different particular values of y, $e^{m_1 x}$, $e^{m_2 x}$, ... $e^{m_n x}$, which satisfy the proposed equation; and therefore its complete integral is

$$y = a_1 e^{m_1 x} + a_2 e^{m_2 x} + \ldots + a_n e^{m_n x}.$$

51. But if any of the roots are equal to one another, as, for instance, $m_1 = m_2$, the value of y becomes

$$y = (a_1 + a_2) e^{m_1 x} + a_3 e^{m_3 x} + \ldots a_n e^{m_n x},$$

which contains only $n-1$ arbitrary constants (because a_1+a_2 can be reckoned only as a single constant), and therefore cannot be the complete integral of the proposed equation. In this case, in order to discover the complete integral, first suppose the two roots m_1, m_2, to be only very nearly equal to one another, so that $m_2 = m_1 + h$, where h is a very small known quantity; then the part of the value of y corresponding to these roots is

$$a_1 e^{m_1 x} + a_2 e^{m_2 x} = e^{m_1 x}(a_1 + a_2 e^{hx}) = e^{m_1 x}\left(a_1 + a_2 + a_2 \frac{hx}{1} + a_2 \frac{h^2 x^2}{1 \cdot 2} + \&c.\right)$$

$$= e^{m_1 x}\left(c_1 + c_2 x + c_2 h \frac{x^2}{2} + \&c.\right),$$

replacing the constants $a_1 + a_2$, and $a_2 h$, by c_1, c_2, respectively. Now let $h = 0$, then this becomes $e^{m_1 x}(c_1 + c_2 x)$; and the complete integral consequently is

$$y = (c_1 + c_2 x)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}.$$

52. Generally, if we suppose r roots of the auxiliary equation to be nearly equal to one another, and therefore to be represented by

$$m_1 + h_1, \quad m_1 + h_2, \ldots m_1 + h_r, \quad \text{where } h_1, h_2, \ldots h_r$$

are very small quantities, the complete integral takes the form

$$y = e^{m_1 x}\left\{a_1 e^{h_1 x} + a_2 e^{h_2 x} + \ldots + a_r e^{h_r x}\right\} + a_{r+1} e^{m_{r+1} x} + \&c.,$$

or, expanding $e^{h_1 x}$, $e^{h_2 x}$, &c.,

$$y = e^{m_1 x}\left\{\Sigma(a) + \Sigma(ah) \cdot x + \frac{\Sigma(ah^2)}{1 \cdot 2} x^2 + \ldots\right.$$

$$\left. + \frac{\Sigma(ah^{r-1})}{\lfloor r-1} x^{r-1} + \frac{\Sigma(ah^r)}{\lfloor r} x^r + \ldots\right\}$$

$$+ a_{r+1} e^{m_{r+1} x} + \ldots$$

or, replacing the constants $\Sigma(a)$, $\Sigma(ah)$, ... $\frac{\Sigma(ah^{r-1})}{\lfloor r-1}$, by $c_1, c_2 \dots c_r$,

$$y = e^{m_1 x}(c_1 + c_2 x + \dots + c_r x^{r-1} + \text{terms involving } h_1, h_2, \text{ \&c.})$$

Now let $h_1 = 0 = h_2 = \dots = h_r$, in which case the auxiliary equation has r roots each $= m_1$; then the solution becomes

$$y = e^{m_1 x}(c_1 + c_2 x + c_3 x^2 + \dots c_r x^{r-1}) + c_{r+1} e^{m_{r+1} x} + \dots + c_n e^{m_n x},$$

which contains n arbitrary constants, and is consequently the complete integral.

53. Of the correctness of the above modification for the case of equal roots, we may assure ourselves by the following reverse process. Let $y = e^{mx} u$, then since

$$d_x^n(uv) = d_x^n v \,.\, u + n d_x^{n-1} v \,.\, d_x u + \frac{n(n-1)}{1.2} d_x^{n-2} v \,.\, d_x^2 u + \text{\&c.}$$

$$d_x^n(e^{mx} u) = e^{mx}\{m^n . u + n m^{n-1} . d_x u + \frac{n(n-1)}{1.2} m^{n-2} d_x^2 u + \text{\&c.}\}$$

$$= e^{mx}(m + d_x)^n u,$$

separating the symbol of operation from that of quantity.

Hence the first member of the equation becomes

$$e^{mx}(m+d_x)^n u + p_1 e^{mx}(m+d_x)^{n-1} u + p_2 e^{mx}(m+d_x)^{n-2} u + \text{\&c.} + p_n e^{mx} u,$$

$$= e^{mx} f(m + d_x)\, u$$

$$= e^{mx}\{f(m)u + \frac{1}{1} f'(m)\, d_x u + \frac{1}{1.2} f''(m)\, d_x^2 u + \dots + \frac{1}{\lfloor n} f^{(n)}(m) d_x^n u\}.$$

Now let $m = m_1$, then since $f(m) = 0$ has r roots equal to m_1, each of the quantities $f(m)$, $f'(m)$, ...$f^{(r-1)}(m)$ becomes equal to zero, and the first r terms vanish; and if

$$u = a_1 + a_2 x + a_3 x^2 + \dots + a_r x^{r-1},$$

then $d_x^r u$, $d_x^{r+1} u$, ... $d_x^n u$ likewise become zero, and all the remaining terms vanish. Hence for each group of r equal roots in the auxiliary equation, there will be in the value of y a term of the form

$$e^{m_1 x}(a_1 + a_2 x + a_3 x^2 + \ldots + a_r x^{r-1}).$$

54. Let $h \pm k\sqrt{-1}$ be a pair of imaginary roots in the auxiliary equation, then the corresponding terms in the value of y will be

$$Ce^{hx + kx\sqrt{-1}} + C'e^{hx + kx\sqrt{-1}},$$

or $e^{hx}\{C(\cos kx + \sqrt{-1}\sin kx) + C'(\cos kx - \sqrt{-1}\sin kx)\}$

$= e^{hx}\{c_1 \cos kx + c_2 \sin kx\}$, (changing the arbitrary constants);

or $= \sqrt{c_1^2 + c_2^2}\, e^{hx}\left\{\dfrac{c_1}{\sqrt{c_1^2 + c_2^2}}\cos kx + \dfrac{c_2}{\sqrt{c_1^2 + c_2^2}}\sin kx\right\}$

$= \beta e^{hx}\cos(kx + a)$,

again changing the constants by putting

$$\sqrt{c_1^2 + c_2^2} = \beta, \text{ and } -\tan a = \frac{c_2}{c_1}.$$

And if there should be r pairs of imaginary roots in the auxiliary equation, equal to $h \pm k\sqrt{-1}$, the corresponding part of the value of y will be

$$e^{hx}(a_1 + a_2 x + \ldots a_r x^{r-1})\,(\cos kx + \sqrt{-1}\sin kx)$$

$$+ e^{hx}(b_1 + b_2 x + \ldots b_r x^{r-1})\,(\cos kx - \sqrt{-1}\sin kx),$$

or, changing the arbitrary constants,

$$e^{hx}(a_1 + a_2 x + \ldots a_r x^{r-1})\cos kx$$

$$+ e^{hx}(\beta_1 + \beta_2 x + \ldots \beta_r x^{r-1})\sin kx.$$

55. We shall now give some examples of integrating linear equations with constant coefficients.

Ex. 1. $d_x^2 y + d_x y - 2y = 0$.

Let $y = e^{mx}$, $\therefore\ m^2 e^{mx} + m e^{mx} - 2e^{mx} = 0$,

$$\text{or}\ m^2 + m - 2 = 0; \quad \therefore\ m = 1, \text{ or } -2;$$

$$\therefore\ y = C_1 e^x + C_2 e^{-2x}.$$

Ex. 2. $a^2 d_x^4 y = d_x^2 y$, $\quad y = C_1 e^{\frac{x}{a}} + C_2 e^{-\frac{x}{a}} + C_3 x + C_4$.

Ex. 3. $d_x^2 y + n^2 y = 0$.

Let $y = e^{mx}$; $\quad \therefore\ m^2 + n^2 = 0$, or $m = \pm n\sqrt{-1}$;

$$\begin{aligned}
\therefore\ y &= C_1 e^{nx\sqrt{-1}} + C_2 e^{-nx\sqrt{-1}} \\
&= C_1 (\cos nx + \sqrt{-1}\sin nx) \\
&\qquad + C_2 (\cos nx - \sqrt{-1}\sin nx) \\
&= (C_1 + C_2)\cos nx + (C_1 - C_2)\sqrt{-1}\sin nx \\
&= a_1 \cos nx + a_2 \sin nx, \text{ (changing the constants)} \\
&= \beta(\cos nx \cos a - \sin nx \sin a) = \beta\cos(nx + a).
\end{aligned}$$

This equation, $d_x^2 y + n^2 y = 0$, of which the solution is

$$y = a_1 \cos nx + a_2 \sin nx, \quad \text{or} \quad y = \beta\cos(nx + a),$$

is of very frequent occurrence. Hence also the solution of

Ex. 4. $d_x^2 y + n^2 y + ax + b = 0$, or

$$d_x^2\left(y + \frac{ax}{n^2} + \frac{b}{n^2}\right) + n^2\left(y + \frac{ax}{n^2} + \frac{b}{n^2}\right) = 0, \text{ is}$$

$$y + \frac{ax}{n^2} + \frac{b}{n^2} = \beta\cos(nx + a).$$

Ex. 5. $d_x^2 y - 2md_x y + (m^2 + n^2) y = 0;$

$y = e^{kx};\quad \therefore\ k^2 - 2km + m^2 + n^2 = 0;\quad \therefore\ k = m \pm n\sqrt{-1};$

$$\therefore\ y = e^{mx}(C\cos nx + C'\sin nx).$$

Ex. 6. $d_x^2 y + 2md_x y + n^2 y = 0;$

$$y = e^{-mx}\beta\cos(x\sqrt{n^2 - m^2} + a),\quad (n > m).$$

Ex. 7. $d_x^4 y - 2d_x^3 y + 2d_x^2 y - 2d_x y + y = 0;$

the equation for determining m is

$$(m^2 + 1)(m - 1)^2 = 0;$$

$$y = e^x(a_1 + a_2 x) + \beta\cos(x + a).$$

Ex. 8. $d_x^4 y + 2n^2 d_x^2 y + n^4 y = 0;$

$$\therefore\ y = (a + a_1 x)\cos nx + (b + b_1 x)\sin nx.$$

Ex. 9. $d_x^4 y + 4md_x^3 y + 2(n^2 + 3m^2) d_x^2 y$

$$+ 4m(m^2 + n^2) d_x y + (m^2 + n^2)^2 y = 0;$$

the equation for determining k, when $y = e^{kx}$, is

$$\{(k + m)^2 + n^2\}^2 = 0;$$

$$\therefore\ y = e^{-mx}(a + a_1 x)\cos nx + e^{-mx}(b + b_1 x)\sin nx.$$

Ex. 10. $d_x^4 y - 4d_x^3 y + 6d_x^2 y - 4d_x y + y = 0;$

$$y = e^x(A + Bx + Cx^2 + Dx^3).$$

56. To integrate the linear equation of the n^{th} order with variable coefficients,

$$d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y \ldots + p_n y = X.$$

Let u_1, u_2, ... u_n be the n particular values of u which satisfy the equation

$$d_x^n u + p_1 d_x^{n-1} u + p_2 d_x^{n-2} u + \ldots + p_n u = 0, \quad (1)$$

which coincides with the proposed if we suppose its second member to become zero; then, as already proved,

$$u = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n.$$

If we now divide both sides by u_1 and differentiate, we shall eliminate c_1; next dividing both sides by the coefficient of c_2, which suppose v_1, and differentiating, we shall eliminate c_2; again dividing by v_2, the coefficient of c_3, and differentiating, we shall eliminate c_3; and proceeding in this manner till all the constants are eliminated, our final result will be of the form (where each d_x affects the whole of the expression after it)

$$d_x \frac{1}{v_{n-1}} d_x \frac{1}{v_{n-2}} \ldots d_x \frac{1}{v_1} d_x \frac{u}{u_1} = 0,$$

in which, the coefficient of $d_x^n u$ is evidently $\dfrac{1}{v_{n-1} v_{n-2} \ldots v_1 u_1}$; therefore dividing by this, so as to make the coefficient of $d_x^n u$ unity, we get

$$v_{n-1} v_{n-2} \ldots v_1 u_1 d_x \frac{1}{v_{n-1}} d_x \frac{1}{v_{n-2}} \ldots d_x \frac{1}{v_1} d_x \frac{u}{u_1},$$

which must be equivalent to the first member of equation (1). Therefore the same expression, only with y instead of u, must be equivalent to the first member of the proposed equation, and therefore equal to X. Hence equating these equals, and reversing the operations, we find the integral of the proposed equation,

$$y = u_1 \int_x v_1 \int_x v_2 \ldots \int_x v_{n-1} \int_x \frac{X}{u_1 v_1 v_2 \ldots v_{n-1}};$$

(each symbol of integration affecting the whole of the expression which follows it), which is a general formula for deducing the solution of a complete linear equation of the n^{th} order, from the solution of the same equation when the term independent of y is supposed to become zero.

57. We shall now give some applications of this method.

Ex. 1. $d_x^2 y + 4 d_x y + 4y = ax^2$.

If we suppose u to denote the value of y in this equation when $a = 0$, since the roots of the auxiliary equation are -2, -2,

$$u = e^{-2x}(c_1 + c_2 x)\,; \quad \therefore\ d_x^2 e^{2x} u = 0,$$

in which expression the coefficient of $d_x^2 u$ is e^{2x}, therefore dividing by this coefficient, and changing u into y, we get

$$e^{-2x} d_x^2 e^{2x} y = ax^2\,;$$

$$\therefore\ y = e^{-2x} \textstyle\int_x^2 (ax^2 e^{2x}) = a\left(\frac{x^2}{4} - \frac{x}{2} + \frac{3}{8}\right) + e^{-2x}(c_1 + c_2 x).$$

Ex. 2. $d_x^2 y - 2m d_x y + (m^2 + n^2) y = X$.

If we suppose u to be the value of y in this equation when $X = 0$, we have

$$u = C_1 e^{mx} \cos nx + C_2 e^{mx} \sin nx\,;$$

$\therefore$ dividing by $e^{mx} \cos nx$, and differentiating, we find

$$d_x e^{-mx} \sec nx \,.\, u = C_2 n \sec^2 nx\,;$$

$$\therefore\ d_x \cos^2 nx\, d_x e^{-mx} \sec nx \,.\, u = 0,$$

in which expression the coefficient of $d_x^2 u$ is $e^{-mx} \cos nx$, therefore dividing by this coefficient and changing u into y, we get

$$e^{mx} \sec nx\, d_x \cos^2 nx\, d_x e^{-mx} \sec nx \,.\, y = X\,;$$

$$\therefore\ y = e^{mx} \cos nx \textstyle\int_x \sec^2 nx \int_x e^{-mx} \cos nx\, X,$$

or, if we suppose the constants to be added after each integration,

$$y = e^{mx}(C \cos nx + C' \sin nx) + e^{mx} \cos nx \textstyle\int_x \sec^2 nx \int_x e^{-mx} \cos nx\, X.$$

Hence the solution of $d_x^2 y + n^2 y = X$ is

$$y = C\cos nx + C'\sin nx + \cos nx \int_x \sec^2 nx \int_x \cos nx\, X.$$

Ex. 3. Let $X = A\cos(mx + a) + B\cos(nx + \beta)$,

then $\int_x \cos nx\, X$

$$= \frac{A}{m^2 - n^2}\left\{m\sin(mx + a)\cos nx - n\cos(mx + a)\sin nx\right\}$$

$$+ \frac{B}{2}\left\{\frac{1}{2n}\sin(2nx + \beta) + x\cos\beta\right\};$$

$$\therefore \int_x \sec^2 nx \int_x \cos nx\, X = -\frac{A}{m^2 - n^2}\cdot\frac{\cos(mx + a)}{\cos nx}$$

$$+ \frac{B}{2}\left\{\frac{x\sin(nx + \beta)}{n\cos nx} - \frac{\sin\beta}{2n^2}\tan nx\right\};$$

$$\therefore\quad y = C\cos nx + C'\sin nx + \frac{A}{n^2 - m^2}\cos(mx + a)$$

$$+ \frac{B}{2n}x\sin(nx + \beta).$$

Ex. 4. $d_x^4 y - 10 d_x^3 y + 62 d_x^2 y - 210 d_x y + 261 y = e^x$.

The roots of the auxiliary equation are

$$2 \pm 5\sqrt{-1},\quad 3,\quad 3;$$

$$\therefore\quad u = (a + a_1 x)e^{3x} + e^{2x}(b\cos 5x + c\sin 5x);$$

$$\therefore\quad d^3(e^{-3x}u) = -e^{-x}\left\{(24b + 10c)\cos 5x + (24c + 10b)\sin 5x\right\};$$

$$\therefore\quad d_x \cos^2 5x\, d_x e^x \sec 5x\, d^2(e^{-3x}u) = 0;$$

$$\therefore\quad e^{2x}\sec 5x\, d_x \cos^2 5x\, d_x e^x \sec 5x\, d_x^2(e^{-3x}y) = e^x;$$

$$\therefore\quad \cos^2 5x\, d_x(\qquad) = e^{-x}\frac{-\cos 5x + 5\sin 5x}{26};$$

$$\therefore \quad d_x e^x \sec 5x\, d_x^2 (e^{-3x} y) = \frac{1}{26}\left(-\frac{e^{-x}}{\cos 5x} + 5e^{-x}\frac{\sin 5x}{\cos^2 5x}\right);$$

$$\therefore \quad e^x \sec 5x\, d_x^2 (e^{-3x} y) = \frac{1}{26}\frac{e^{-x}}{\cos 5x};$$

$$\therefore \quad d_x^2 (e^{-3x} y) = \frac{1}{26} e^{-2x}, \qquad y = \frac{e^x}{104};$$

$$\therefore \quad y = \frac{e^x}{104} + (a + a_1 x) e^{3x} + e^{2x} (b \cos 5x + c \sin 5x).$$

Ex. 5. $\quad d_x^4 y + p_1 d_x^3 y + p_2 d_x^2 y + p_3 d_x y + p_4 y = X;$

where the coefficients are such that the auxiliary equation

$$k^4 + p_1 k^3 + p_2 k^2 + p_3 k + p_4 = \{(k+m)^2 + n^2\}^2 = 0,$$

has two pairs of imaginary roots. Then the solution of

$$d_x^4 u + p_1 d_x^3 u + p_2 d_x^2 u + p_3 d_x u + p_4 u = 0, \quad \text{is}$$

$$e^{mx} u = (a + a^1 x) \cos nx + (b + b^1 x) \sin nx;$$

$$\therefore \quad d_x (\sec nx\, e^{mx} u) = a^1 + b^1 \tan nx + n(b + b^1 x) \sec^2 nx;$$

$$\therefore \quad \cos^2 nx\, d_x (\qquad) = \frac{a^1}{2}(1 + \cos 2nx) + \frac{b^1}{2}\sin 2nx$$

$$+ n(b + b^1 x);$$

$$\therefore \quad d_x (\qquad) = -na^1 \sin 2nx + nb^1 (1 + \cos 2nx);$$

$$\therefore \quad \sec^2 nx\, d_x (\qquad) = -2na^1 \tan nx + 2nb^1;$$

$$\therefore \quad d_x \sec^2 nx\, d_x (\qquad) = -2n^2 a^1 \sec^2 nx;$$

$$\therefore \quad d_x \cos^2 nx\, d_x \sec^2 nx\, d_x (\qquad) = 0;$$

and as the coefficient of $d_x^4 u$ is $e^{mx}\cos nx$, dividing by this coefficient and changing u into y, we get

$$e^{-mx}\sec nx\, d_x \cos^2 nx\, d_x \sec^2 nx\, d_x \cos^2 nx\, d_x \sec nx\, e^{mx} y = X;$$

$$\therefore\ e^{mx} y = \cos nx \int_x \sec^2 nx \int_x \cos^2 nx \int_x \sec^2 nx \int_x \cos nx\, e^{mx} X;$$

or, if we suppose a constant to be added after each integration, we may add to this value of $e^{mx}y$, the terms

$$(a + a^1 x)\cos nx + (b + b^1 x)\sin nx.$$

Ex. 6. Suppose the linear equation to have constant coefficients, and let $-a_1$, $-a_2, \ldots -a_n$ be the roots of its auxiliary equation, so that

$$m^n + p_1 m^{n-1} + p_2 m^{n-2} + \ldots + p_n = (m + a_1)(m + a_2)\ldots(m + a_n),$$

$$\text{then}\ \ u = c_1 e^{-a_1 x} + c_2 e^{-a_2 x} + c_3 e^{-a_3 x} + \ldots + c_n e^{-a_n x};$$

$$d_x(e^{a_1 x} u) = c_2 (a_1 - a_2)\, e^{(a_1 - a_2)x}$$

$$+ c_3 (a_1 - a_3)\, e^{(a_1 - a_3)x} + \ldots + c_n (a_1 - a_n)\, e^{(a_1 - a_n)x};$$

$$d_x e^{(a_2 - a_1)x} d_x (e^{a_1 x} u) = c_3 (a_2 - a_3)(a_1 - a_3)\, e^{(a_2 - a_3)x} + \ldots$$

$$+ c_n (a_2 - a_n)(a_1 - a_n)\, e^{(a_2 - a_n)x},$$

&c. = &c.

$$d_x e^{(a_n - a_{n-1})x} d_x e^{(a_{n-1} - a_{n-2})x} \ldots d_x e^{(a_2 - a_1)x} d_x (e^{a_1 x} u) = 0;$$

therefore, dividing by $e^{a_n x}$, which is the coefficient of $d_x^n u$, and replacing u by y, we get

$$e^{-a_n x} d_x e^{(a_n - a_{n-1})x} \ldots d_x e^{(a_2 - a_1)x} d_x (e^{a_1 x} y) = X;$$

$$\therefore\ y = e^{-a_1 x} \int_x e^{(a_1 - a_2)x} \int_x e^{(a_2 - a_3)x} \ldots \int_x e^{(a_{n-1} - a_n)x} \int e^{a_n x} X.$$

Method of Parameters.

58. There is also another mode of integrating linear equations, which deserves to be mentioned, called the method of Parameters, which we shall now explain. It may be stated thus. The complete integral of the linear equation of the n^{th} order will be of the form

$$y = v_1 u_1 + v_2 u_2 + \ldots + v_n u_n,$$

where $u_1, u_2, \ldots u_n$ are the n particular integrals of the equation when the term independent of y becomes zero, and $v_1, v_2, \ldots v_n$ are functions of x, determined by equations of the form

$$v_1 = \int_x f_1(x) + C_1.$$

Let the proposed equation be

$$d_x^n y + p_1 d_x^{n-1} y + \ldots + p_n y = X,$$

and suppose $\quad y = v_1 u_1 + v_2 u_2 + \ldots v_n u_n = \Sigma(vu),$

and as we have made only one assumption respecting the n independent quantities $v_1, v_2 \ldots v_n$, we may make $n-1$ more;

$$\text{now} \quad d_x y = \Sigma(v d_x u) + \Sigma(u d_x v);$$

$$\therefore \quad d_x y = \Sigma(v d_x u),$$

putting for the first of our additional assumptions

$$\Sigma(u d_x v) = 0, \quad (1);$$

similarly $\; d_x^2 y = \Sigma(v d_x^2 u), \quad$ putting $\; \Sigma(d_x u \, d_x v) = 0, \quad (2)$

&c. = &c.

and $\; d_x^n y = \Sigma(v d_x^n u) + X, \quad$ putting $\; \Sigma(d_x^{n-1} u d_x v) = X, \; (n).$

Substitute these values for y, $d_x y$, $d_x^2 y, \ldots d_x^n y$, in the proposed equation, and the first member becomes

$$v_1 (d_x^n u_1 + p_1 d_x^{n-1} u_1 + \ldots + p_n u_1) + v_2 (d_x^n u_2 + p_1 d_x^{n-1} u_2 + \ldots + p_n u_2)$$

$$+ \text{\&c.} + v_n (d_x^n u_n + p_1 d_x^{n-1} u_n + \ldots + p_n u_n) + X,$$

which manifestly reduces itself to X, since each of the quantities within brackets is equal to zero; hence if the parameters $v_1, v_2, \ldots v_n$ be determined subject to the above n equations of condition, the assumed value of y will satisfy the equation; and since there are n equations in which the n quantities $d_x v_1$, $d_x v_2$, ... $d_x v_n$ enter to the first degree, each of the parameters v_1, v_2, &c. will be determined by an equation of the form $d_x v_1 = f_1(x)$, and so n arbitrary constants will be introduced.

Ex. 1. $d_x^2 y + n^2 y = \sec nx$.

The solution of $d_x^2 y + n^2 y = 0$ is $y = a_1 \cos nx + a_2 \sin nx$.

Let $\therefore\ y = v_1 \cos nx + v_2 \sin nx$;

$$\therefore\ d_x y = -v_1 n \sin nx + v_2 n \cos nx,$$

making $\quad d_x v_1 \cos nx + d_x v_2 \sin nx = 0.$

$$d_x^2 y = -v_1 n^2 \cos nx - v_2 n^2 \sin nx + \sec nx,$$

making $\quad -d_x v_1 n \sin nx + d_x v_2 n \cos nx = \sec nx$;

$$\therefore\ d_x v_1 = -\frac{1}{n}\frac{\sin nx}{\cos nx}, \quad \text{or} \quad v_1 = \frac{1}{n^2} \log \cos nx + C_1.$$

$$d_x v_2 = \frac{1}{n}, \qquad \text{or} \quad v_2 = \frac{x}{n} + C_2;$$

$$\therefore\ y = \left(\frac{1}{n^2} \log \cos nx + C_1\right) \cos nx + \left(\frac{x}{n} + C_2\right) \sin nx.$$

Ex. 2. $d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y + \dots + p_n y = X$,

where the coefficients are constant, and the auxiliary equation, $M = 0$, has none of its roots $-a_1$, $-a_2$, ... $-a_n$, equal to one another.

Here $u_1 = e^{-a_1 x}$, $u_2 = e^{-a_2 x}$, ... $u_n = e^{-a_n x}$;

and in order to determine the parameters v_1, v_2, &c. assume

$$d_x v_1 = c_1 e^{a_1 x} X,$$
$$d_x v_2 = c_2 e^{a_2 x} X,$$
$$\dots\dots\dots\dots\dots$$
$$d_x v_n = c_n e^{a_n x} X,$$

then, by substitution, the n equations of condition become

$$\Sigma(c) = 0, \quad \Sigma(ac) = 0, \quad \Sigma(a^2 c) = 0, \quad \dots \Sigma(a^{n-1} c) = 1;$$

and from these equations, we have to determine each of the constants $c_1, c_2, \dots c_n$, by eliminating the rest; for this purpose multiply them in the reverse order by the indeterminate coefficients $1, q_1, q_2, \dots q_{n-1}$, and take the sum; then

$$c_1 (a_1^{n-1} + q_1 a_1^{n-2} + q_2 a_1^{n-3} + \dots + q_{n-1})$$
$$+ c_2 (a_2^{n-1} + q_1 a_2^{n-2} + q_2 a_2^{n-3} + \dots + q_{n-1})$$
$$+ \dots + c_n (a_n^{n-1} + q_1 a_n^{n-2} + \dots + q_{n-1}) = 1.$$

Now assume $q_1, q_2, \dots q_{n-1}$ so that the coefficients of $c_2, c_3, \dots c_n$ may each equal zero; then $a_2, a_3, \dots a_n$ are the roots of the equation

$$a^{n-1} + q_1 a^{n-2} + q_2 a^{n-3} + \dots + q_{n-1} = 0,$$

therefore, replacing the coefficient of c_1 by the product of its simple factors,

$$c_1 (a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n) = 1, \quad \text{or} \quad c_1 = \frac{1}{d_{m=a_1} M};$$

$$\therefore \quad v_1 = \frac{1}{d_{m=a_1} M} \int_x e^{a_1 x} X; \text{ similarly, } v_2 = \frac{1}{d_{m=a_2} M} \int_x e^{a_2 x} X, \text{ \&c.};$$

$$\therefore \quad y = {}^1\Sigma^n \left\{ \frac{e^{-ax}}{d_{m=a} M} \int_x e^{ax} X \right\}.$$

This method evidently fails when $M = 0$ has equal roots, since one or more of the quantities $d_{m=a_1} M$, $d_{m=a_2} M$, &c. in that case becomes zero.

Ex. 3. $$d_x^4 y - y = e^x.$$

Here

$$M = m^4 - 1 = (m+1)(m-1)(m+\sqrt{-1})(m-\sqrt{-1}) = 0;$$

therefore the four values of $\int_x e^{ax} X$ are

$$\frac{e^{2x}}{2} + c_1, \quad x + c_2, \quad \frac{e^{x(1+\sqrt{-1})}}{1+\sqrt{-1}} + c_3, \quad \frac{e^{x(1-\sqrt{-1})}}{1-\sqrt{-1}} + c_4,$$

and the four values of $\dfrac{e^{-ax}}{d_{m=a} M}$ are

$$\frac{e^{-x}}{4}, \quad \frac{e^x}{-4}, \quad \frac{e^{-x\sqrt{-1}}}{-4\sqrt{-1}}, \quad \frac{e^{x\sqrt{-1}}}{4\sqrt{-1}};$$

$$\therefore \quad y = \frac{e^{-x}}{4}\left(\frac{e^{2x}}{2} + c_1\right) + \frac{e^x}{-4}(x + c_2) + \frac{e^{-x\sqrt{-1}}}{-4\sqrt{-1}}\left(\frac{e^{x(1+\sqrt{-1})}}{1+\sqrt{-1}} + c_3\right)$$

$$+ \frac{e^{x\sqrt{-1}}}{4\sqrt{-1}}\left(\frac{e^{x(1-\sqrt{-1})}}{1-\sqrt{-1}} + c_4\right)$$

$$= \frac{e^x}{8} + c_1 e^{-x} - \frac{xe^x}{4} + c_2 e^x + \frac{e^x}{4} + c_3 \cos x + c_4 \sin x$$

$$= \frac{3e^x}{8} - \frac{xe^x}{4} + c_1 e^{-x} + c_2 e^x + c_3 \cos x + c_4 \sin x.$$

Ex. 4. $$d_x^4 y - y = \cos x;$$

the four values of $\int_x e^{ax} X$ are

$$\frac{e^x}{2}(\cos x + \sin x) + c_1, \quad \frac{e^{-x}}{2}(-\cos x + \sin x) + c_2,$$

$$\tfrac{1}{2}(x + \sin x e^{x\sqrt{-1}}) + c_3, \quad \tfrac{1}{2}(x + \sin x e^{-x\sqrt{-1}}) + c_4,$$

and the values of $\dfrac{e^{-ax}}{d_{m=a}M}$ are the same as in the preceding example;

$$\therefore \quad y = \frac{1}{8}(\cos x + \sin x) + \frac{c_1}{4}e^{-x} + \frac{1}{8}(\cos x - \sin x) - \frac{c_2}{4}e^{x}$$

$$- \frac{1}{8\sqrt{-1}}(xe^{-x\sqrt{-1}} + \sin x) - \frac{c_3}{4\sqrt{-1}}e^{-x\sqrt{-1}}$$

$$+ \frac{1}{8\sqrt{-1}}(xe^{x\sqrt{-1}} + \sin x) + \frac{c_4}{4\sqrt{-1}}e^{x\sqrt{-1}}$$

$$= \frac{\cos x}{4} + \frac{x \sin x}{4} + \tfrac{1}{4}(c_1 e^{-x} - c_2 e^{x})$$

$$+ \frac{c_4}{4\sqrt{-1}}(\cos x + \sqrt{-1}\sin x) - \frac{c_3}{4\sqrt{-1}}(\cos x - \sqrt{-1}\sin x)$$

$$= \frac{x \sin x}{4} + a_1 e^{x} + a_2 e^{-x} + a_3 \cos x + a_4 \sin x.$$

59. When the auxiliary equation contains equal roots, we may, by the following theorem, depress the proposed equation to one whose auxiliary equation shall contain only unequal roots, and then complete the solution by variation of parameters.

If $d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y + \ldots + p_n y = X_n$ be a linear equation of the n^{th} order, with constant coefficients; and $-a_1, -a_2, \ldots -a_n$ the roots of its auxiliary equation, so that

$$m^n + p_1 m^{n-1} + p_2 m^{n-2} + \ldots + p_n = (m + a_1)(m + a_2)\ldots(m + a_n);$$

then if we multiply both sides of the proposed by e^{ax}, $-a$ being any root, and integrate, the result will be a linear equation of the $(n-1)^{\text{th}}$ order with the same auxiliary equation, except that it wants the factor $m + a$.

For let $m^{n-1} + q_1 m^{n-2} + \ldots + q_{n-1} = 0$ be this auxiliary equation.

$$\text{Now } d_x e^{ax} (d_x^{n-1} y + q_1 d_x^{n-2} y + \ldots + q_{n-1} y)$$

$$= e^{ax} \{d_x^n y + (q_1 + a)\, d_x^{n-1} y + (q_2 + q_1 a)\, d_x^{n-2} y + \ldots + q_{n-1} a y\}$$

$$= e^{ax} (d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y + \ldots + p_n y) = e^{ax} X_n;$$

$$\therefore \quad d_x^{n-1} y + q_1 d_x^{n-2} y + \ldots + q_{n-1} y = e^{-ax} \textstyle\int_x e^{ax} X_n = X_{n-1},$$

the auxiliary equation being $m^{n-1} + q_1 m^{n-2} + \ldots + q_{n-1} = 0$, cleared of the factor $m + a$; and if this factor should occur r times, then repeating this process r times, we should get

$$d_x^{n-r} y + t_1 d_x^{n-r-1} y + \ldots + t_{n-r} y = e^{-ax} \textstyle\int_x^r e^{ax} X_n,$$

whose auxiliary equation differs from $M = 0$, only in wanting the factor $(m + a)^r$.

60. If we know a particular integral of a linear equation of any order that has no term independent of y, we may reduce it to another equation of the same kind, of the order immediately inferior.

Let $y = u$ be a particular integral of $f(d_x)\, y = 0$,

$$\text{or } d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y + \ldots + p_n y = 0, \quad (1)$$

and assume $y = u \int_x z$, z being a new variable, a function of x; then, separating the symbols of operation from those of quantity, we have

$$d_x^n y = (d_x + d_x^1)^n u \textstyle\int_x z,$$

d_x being understood to affect u only, and d_x^1 to affect $\int_x z$ only.

Hence, substituting for the differential coefficients in the proposed, by this formula, and again separating the symbols of operation from those of quantity, we get successively

$$(d_x + d_x^1)^n u \textstyle\int_x z + p_1 (d_x + d_x^1)^{n-1} u \int_x z + \&\text{c.} + p_n u \int_x z$$
$$= f(d_x + d_x^1)\, u \textstyle\int_x z$$

$$= \int_x z \,.\, f(d_x)\, u + \frac{1}{1}\, z f'(d_x)\, u + \frac{1}{1\,.\,2}\, d_x z f''(d_x)\, u + \dots$$

$$+ \frac{1}{\lfloor n-1}\, d_x^{n-2} z f^{n-1}(d_x)\, u + d_x^{n-1} z \,.\, u.$$

But since u is a particular value of y, $f(d_x)\, u = 0$; hence, reversing the order of the terms, and observing that $f^{(r)}(d_x)$ denotes the same function of d_x, that $d_v^r f(v)$ does of v, we get for the depressed equation,

$$u d_x^{n-1} z + (n d_x u + p_1 u)\, d_x^{n-2} z$$

$$+ \left\{\frac{n(n-1)}{1\,.\,2} d_x^2 u + (n-1) p_1 d_x u + p_2 u\right\} d_x^{n-3} z + \dots + f'(d_x) u \,.\, z = 0. \quad (2)$$

Similarly, if we know another particular solution, u_1, of equation (1), then $d_x \left(\frac{u_1}{u}\right)$ will be a value of z in equation (2), and we may depress this equation to another of the same form of the $(n-2)^{\text{th}}$ order; and if we know r particular solutions of equation (1), we may in this way depress it to an equation of the same form of the $(n-r)^{\text{th}}$ order.

61. If $y = u$ be a particular integral of the equation

$$d_x^2 y + P d_x y + Q y = 0,$$

the other particular integral will be $y = u \int_x \frac{e^{-\int_x P}}{u^2}$.

For if we denote the two values of y by u and v, we have

$$d_x^2 u + P d_x u + Q u = 0, \quad d_x^2 v + P d_x v + Q v = 0;$$

$$\therefore\; u d_x^2 v - v d_x^2 u + P(u d_x v - v d_x u) = 0;$$

$$\therefore\; u d_x v - v d_x u = C e^{-\int_x P};$$

$$\therefore\; d_x \left(\frac{v}{u}\right) = \frac{1}{u^2}\, C e^{-\int_x P};$$

$$\therefore\; v = C u \int_x \frac{e^{-\int_x P}}{u^2}.$$

Change of Independent Variable.

62. Besides linear equations with constant coefficients, very few equations of the second and superior orders admit of being integrated; and those only by particular methods. Sometimes the integration of an equation may be facilitated by changing the independent variable.

Ex. $d_x^2 y + a\,(d_x y)^2 + bx\,(d_x y)^3 = 0.$

Make y the independent variable, and consequently for $d_x y$, $d_x^2 y$, write

$$\frac{1}{d_y x}, \quad -\frac{d_y^2 x}{(d_y x)^3},$$

$$\therefore \quad -\frac{d_y^2 x}{(d_y x)^3} + \frac{a}{(d_y x)^2} + \frac{bx}{(d_y x)^3} = 0,$$

$$\text{or} \quad d_y^2 x - a\,d_y x - bx = 0,$$

which is integrable.

$x = \varepsilon^{my}$

63. In the above example there is no difficulty in fixing upon the new independent variable. In other cases we must consider x and y as functions of a third variable t, and substitute the values of $d_x y$, $d_x^2 y$, &c. corresponding to that supposition. When the equation is thus generalized, we must assume for x or y some known function of t, according to the circumstances of the case, so that there may arise for determining the function of t which expresses y or x, a differential equation simpler than the proposed one; between the integral of which and the assumed function, if we eliminate t, we obtain the required relation between x and y.

Ex. 1. $(1 - x^2)\,d_x^2 y - x\,d_x y + n^2 y = 0.$

The generalized equation is

$$(1 - x^2)\,\frac{d_t x\, d_t^2 y - d_t y\, d_t^2 x}{(d_t x)^3} - x\,\frac{d_t y}{d_t x} + n^2 y = 0.$$

Let $x = \cos t$, then $d_t x = -\sin t$, $d_t^2 x = -\cos t$;

therefore, by substitution, $d_t^2 y + n^2 y = 0$, which gives

$$y = A \cos nt + B \sin nt;$$

$$\therefore\ y = A \cos(n \cos^{-1} x) + B \sin(n \cos^{-1} x).$$

Ex. 2. $$d_x^2 y + \frac{A}{a + bx} d_x y + \frac{B}{(a + bx)^2} y = 0.$$

The generalized equation is

$$\frac{d_t x\, d_t^2 y - d_t y\, d_t^2 x}{(d_t x)^3} + \frac{A}{a + bx} \frac{d_t y}{d_t x} + \frac{B}{(a + bx)^2} y = 0.$$

Let $d_t x = a + bx$, or $bt = \log(a + bx)$, or $e^{bt} = a + bx$;

then $d_t^2 x = b\, d_t x$;

therefore, by substitution,

$$d_t^2 y + (A - b)\, d_t y + By = 0;$$

the solution of which is

$$y = c_1 e^{mt} + c_2 e^{m't} = c_1 (a + bx)^{\frac{m}{b}} + c_2 (a + bx)^{\frac{m'}{b}},$$

m and m' being the roots, supposed possible, of

$$m^2 + (A - b) m + B = 0.$$

If the roots be impossible, and of the form $m \pm n\sqrt{-1}$, the solution is

$$y = \beta e^{mt} \cos(nt + a) = \beta (a + bx)^{\frac{m}{b}} \cos\left\{\log(a + bx)^{\frac{n}{b}} + a\right\}.$$

Ex. 3. $$(a + bx)^3 d_x^3 y + A(a + bx)^2 d_x^2 y + B(a + bx)\, d_x y + Cy = f(x).$$

As in the preceding Example,

$$\text{let} \quad a + bx = e^{bt}, \quad \therefore \; d_t x = e^{bt},$$

$$d_x y = e^{-bt} d_t y,$$

$$d_x^2 y = e^{-2bt} (d_t^2 y - b\, d_t y),$$

$$d_x^3 y = e^{-3bt} (d_t^3 y - 3b\, d_t^2 y + 2b^2 d_t y);$$

$\therefore$ by substitution we get

$$d_t^3 y + (A - 3b)\, d_t^2 y + (2b^2 - Ab + B)\, d_t y + Cy = f\left(\frac{e^{bt} - a}{b}\right),$$

a linear equation with constant coefficients.

The same substitution of course succeeds for the equation,

$$(a+bx)^n d_x^n y + A(a+bx)^{n-1} d_x^{n-1} y + \&\text{c.} + K(a+bx)\, d_x y + Ly = f(x),$$

which, by putting $a + bx = z$, may first of all be reduced to the form,

$$z^n d_z^n y + a_1 z^{n-1} d_z^{n-1} y + \&\text{c.} + a_{n-1} z\, d_z y + a_n y = f_1(z);$$

and then to a linear equation with constant coefficients, by the formula

$$z^n d_z^n y = d_t^n y - p_1 d_t^{n-1} y + \&\text{c.} \pm p_{n-1} y,$$

where $z = e^t$, and p_1, p_2, &c. are such that the roots of

$$k^n - p_1 k^{n-1} + p_2 k^{n-2} - \&\text{c.} \ldots \pm p_{n-1} k = 0,$$

are $0, 1, 2, 3 \ldots (n - 1)$.

Ex. 4. $\quad x^2 d_x^2 y + x d_x y - y = \dfrac{ax^2}{x^2 - 1}$,

$$y = \frac{C_1}{x} + C_2 x + \frac{a}{2} + \frac{a}{4}\left(\frac{1}{x} - x\right) \log\left(\frac{x - 1}{x + 1}\right).$$

The substitution $x = e^t$ gives

$$d_t^2 y - y = a + \frac{a}{e^{2t} - 1};$$

$$\therefore\ \ y = C_1 e^{-t} + C_2 e^{t} + \frac{a}{2} + \frac{a}{4}(e^{-t} - e^{t}) \log\left(\frac{e^{t}-1}{e^{t}+1}\right),$$

which agrees with the above, upon writing x for e^t.

Ex. 5. $x^2 d_x^2 y + x d_x y - y = a x^n$,

$$y = \frac{C_1}{x} + C_2 x + \frac{a x^n}{n^2 - 1},$$

Solutions expressed by Definite Integrals. X

64. Sometimes the complete integral of a differential equation may be expressed by a definite integral, as in the following instances.

Ex. 1. $d_x^{n-1} y = xy + m.$

Let $v_1 = a_1 \int_t^{\infty} {}^{0} e^{a_1 t x} . e^{-\frac{t^n}{n}}$, a_1 being a constant quantity;

$$\text{then}\quad d_x^{n-1} v_1 = a_1^n \int_t^{\infty} {}^{0} t^{n-1} e^{a_1 t x} e^{-\frac{t^n}{n}} = a_1^n (1 + x v_1),$$

change a_1 into a_2, a_3, &c., and let v_2, v_3, &c. be the corresponding values of v_1,

$$\therefore\quad d_x^{n-1} v_2 = a_2^n (1 + x v_2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$d_x^{n-1} v_n = a_n^n (1 + x v_n).$$

Now suppose a_1, a_2, ... a_n to be the n^{th} roots of unity; then multiplying each of these equations by an arbitrary constant, and taking the sum,

$$d_x^{n-1} (c_1 v_1 + c_2 v_2 + \dots + c_n v_n) = c_1 + c_2 + \dots$$
$$+ c_n + x (c_1 v_1 + c_2 v_2 + \dots + c_n v_n),$$

$$\text{let}\quad c_1 + c_2 + \dots + c_n = m,$$

then $y = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$ is the complete integral, containing only $n-1$ arbitrary constants by reason of the equation of condition; or it may be written

$$y = {}^0\!\int_t^\infty e^{-\frac{t^n}{n}} \{ce^{tx} + c_1 a e^{atx} + c_2 a^2 e^{a^2 tx} + \ldots + c_{n-1} a^{n-1} e^{a^{n-1} tx}\},$$

a being a primitive root of $k^n - 1 = 0$.

Let $m = 0$, $n = 3$, then $d_x^2 y = xy$, the solution of which is

$$y = {}^0\!\int_t^\infty e^{-\frac{1}{3}t^3 - \frac{1}{2}xt}\left\{Be^{\frac{3tx}{2}} + \tfrac{1}{2}(B + A\sqrt{3}) \cos\frac{\sqrt{3}tx}{2}\right.$$
$$\left. - \tfrac{1}{2}(A + B\sqrt{3}) \sin\frac{\sqrt{3}tx}{2}\right\}.$$

Let $m = 0$, $n = 2$, then $d_x y = xy$, and

$$y = C\,{}^0\!\int_t^\infty e^{-\frac{t^2}{2}}(e^{xt} + e^{-xt}) = \frac{2C}{\sqrt{-1}}\,{}^0\!\int_z^\infty e^{\frac{z^2}{2}} \cos xz, \text{ putting } t = z\sqrt{-1},$$

$$= \frac{2C}{\sqrt{-1}} \cdot \frac{\sqrt{\pi} e^{\frac{x^2}{2}}}{\sqrt{-2}} = C\sqrt{2\pi} e^{\frac{x^2}{2}}, \quad \text{(Integ. Cal. Art. 112).}$$

which agrees with the result obtained by direct integration.

If the more general form were proposed,

$$d_x^{n-1} y = (a + \beta x) y,$$

assume $a + \beta x = \beta^{\frac{n-1}{n}} t$; $\quad \therefore \; d_x y = d_t y \, d_x t = \beta^{\frac{1}{n}} d_t y$,

$$\text{and} \quad d_x^{n-1} y = \beta^{\frac{n-1}{n}} d_t^{n-1} y = \beta^{\frac{n-1}{n}} t y;$$

$$\therefore \; d_t^{n-1} y = ty;$$

and to this form may the still more general case

$$d_x^{n-1} y = axy + bx + cy + e$$

be reduced. (The above solution is taken from Crelle's Journal, Vol. x.)

Ex. 2. $d_x^2 y + \left(n^2 - \dfrac{a}{x^2}\right) y = 0$*.

Let $y = \dfrac{u}{x^m}$ where $m(m+1) = a$, and suppose a to be such that m is a whole number; then

$$x(d_x^2 u + n^2 u) = 2m d_x u. \quad (1).$$

Differentiate this 2, 4, &c. times successively, and add each result to the preceding multiplied by n^2, in doing which the symbols of operation and quantity may be conveniently separated, then

$$x d_x^2 (d_x^2 + n^2) u + 2 d_x (d_x^2 + n^2) u = 2m d_x^3 u,$$

$$\text{and} \quad x n^2 (d_x^2 + n^2) u = 2m n^2 d_x u;$$

$$\therefore \quad x (d_x^2 + n^2)^2 u = 2m d_x (d_x^2 + n^2) u - 2 d_x (d_x^2 + n^2) u$$

$$= 2(m-1) d_x (d_x^2 + n^2) u;$$

$$\text{similarly,} \quad x (d_x^2 + n^2)^3 u = 2(m-2) d_x (d_x^2 + n^2)^2 u,$$

and repeating the process m times, the second member becomes zero, and

$$(d_x^2 + n^2)^{m+1} u = 0.$$

Hence we have, for the determination of u, a linear equation with constant coefficients, whose auxiliary equation is

$$(k^2 + n^2)^{m+1} = 0;$$

$$\therefore \quad u = (a + a_1 x + a_2 x^2 + \ldots + a_m x^m) \cos nx$$

$$+ (b + b_1 x + b_2 x^2 + \ldots b_m x^m) \sin nx,$$

* This equation, when $a = 6$, presents itself in investigations relative to the figure of the Earth; the solution in the forms here noticed was given by Mr Gaskin (Cambridge Problems, 1839).

this gives the form of the solution; the arbitrary constants must be reduced to the proper number, in any case, by actual substitution; thus let $m = 2$, so that $a = 6$; then

$$u = (a + a_1 x + a_2 x^2) \cos nx + (b + b_1 x + b_2 x^2) \sin nx,$$

and substituting this value in (1) in order to reduce the constants, we find

$$x^2 y = u = a \left\{\cos nx \left(1 - \frac{n^2 x^2}{3}\right) + nx \sin nx\right\}$$

$$+ b \left\{\sin nx \left(1 - \frac{n^2 x^2}{3}\right) - nx \cos nx\right\},$$

for the complete integral of

$$d_x^2 y + \left(n^2 - \frac{6}{x^2}\right) y = 0.$$

But when in the equation $m(m+1) = a$, m is a fraction,

make $y = x^{m+1} u$, then $x(d_x^2 u + n^2 u) + 2(m+1) d_x u = 0$;

the complete integral of which is

$$u = \beta\, {}^{-n}\!\int_t^n (t^2 - n^2)^m \cos(xt + a),$$

a and β being the arbitrary constants; for this gives

$$d_x u = \beta\, {}^{-n}\!\int_t^n (t^2 - n^2)^m (-t) \sin(xt + a)$$

$$= \frac{\beta x}{2(m+1)}\, {}^{-n}\!\int_t^n (t^2 - n^2)^{m+1} \cos(xt + a);$$

$$d_x^2 u = \beta\, {}^{-n}\!\int_t^n (t^2 - n^2)^m (-t^2) \cos(xt + a)$$

$$= \beta\, {}^{-n}\!\int_t^n - (t^2 - n^2)^m (t^2 - n^2 + n^2) \cos(xt + a);$$

$$\therefore\ d_x^2 u + n^2 u = -\beta \,{}^{-n}\!\int_t^{n} (t^2 - n^2)^{m+1} \cos(xt + a) = -\frac{2(m+1)}{x} d_x u.$$

Similarly, when m is an integer, it will appear by substitution that

$$u = \beta d_{t=n^2}^{m} \left\{ \frac{\cos(x\sqrt{t} + a)}{\sqrt{t}} \right\}$$

is the complete integral of $x(d_x^2 u + n^2 u) = 2md_x u$, the reduced equation when m is an integer.

Simultaneous Equations.

65. In these equations, which sometimes occur in the higher parts of Dynamics, instead of one equation between x, y, and the differential coefficients of y with respect to x, being given to determine the relation between x and y; we have two equations containing x, y, t (of which x and y are considered as functions) and the differential coefficients of x and y relative to t, to find that relation.

66. To integrate the simultaneous equations of the first order,

$$ax + by + d_t x = T, \qquad a'x + b'y + d_t y = T',$$

T and T' denoting functions of t. Multiplying the latter by an indeterminate quantity m, and adding it to the former, we get

$$ax + by + m(a'x + b'y) + d_t(x + my) = T + mT',$$

$$\text{or}\quad d_t(x + my) + (a + ma')\left(x + \frac{b + mb'}{a + ma'} y\right) = T + mT'.$$

Let $\dfrac{b + mb'}{a + ma'} = m$, which will give two values of m, m_1 and m_2; then the equation, under this condition, becomes a linear equation of the first order; and we obtain by integration

$$(x + my)\, e^{(a + ma')t} = \int_t (T + mT')\, e^{(a + ma')t};$$

and by substituting successively the two values of m, we obtain two primitive equations, which will furnish values of x and y in terms of t, and the relation between x and y, if t be eliminated. If the two values of m are equal, we shall obtain only one equation between x, y, and t; but if this can be solved with respect to x or y, and we substitute the value so found in one of the given equations, we shall obtain a second relation either between x and t, or between y and t; and then t may be eliminated as before.

Ex. 1. $5x - 2y + d_t x = e^t, \qquad 6y - x + d_t y = e^{2t};$

$$\therefore \quad (5-m)\,x + (-2+6m)\,y + d_t\,(x+my) = e^t + me^{2t},$$

let $\dfrac{-2+6m}{5-m} = m$, or $m^2 + m - 2 = 0$, or $m = 1$, or -2;

$$\therefore \quad d_t\,(x+y) + 4\,(x+y) = e^t + e^{2t};$$

$$\therefore \quad x + y = \frac{e^t}{5} + \frac{e^{2t}}{6} + Ce^{-4t};$$

$$\text{similarly} \quad x - 2y = \frac{e^t}{8} - \frac{2}{9}e^{2t} + C_1 e^{-7t},$$

which determine x and y in terms of t.

Ex. 2. $d_t x + 5x + y = e^t, \qquad d_t y + 3y - x = e^{2t};$

$$\therefore \quad d_t\,(x+my) + (5-m)\,x + (1+3m)\,y = e^t + me^{2t};$$

$$\therefore \quad \frac{1+3m}{5-m} = m, \quad \text{or} \quad 1 - 2m + m^2 = (1-m)^2 = 0.$$

Hence the values of m are each $= 1$, and integrating, we find

$$x + y = C_1 e^{-4t} + \frac{1}{5}e^t + \frac{1}{6}e^{2t}.$$

By means of this, eliminate y from the first equation, and we get

$$d_t x + 4x = \frac{4}{5}e^t - \frac{e^{2t}}{6} - C_1 e^{-4t};$$

$$\therefore \quad x = \frac{4}{25}e^t - \frac{1}{36}e^{2t} - C_1 t e^{-4t} + C_2 e^{-4t},$$

$$\text{and} \quad y = \frac{1}{25}e^t + \frac{7}{36}e^{2t} + C_1(1+t)e^{-4t} - C_2 e^{-4t}.$$

The more general form

$$ax + by + Ad_t x + Bd_t y = T, \quad a'x + b'y + A'd_t x + B'd_t y = T',$$

may evidently be reduced to the above by successively eliminating $d_t y$, $d_t x$.

67. To integrate the simultaneous equation of the second order,

$$ax + by + c + d_t^2 x = 0, \quad a'x + b'y + c' + d_t^2 y = 0.$$

Multiplying the latter by an indeterminate quantity m, and adding it to the former, we get

$$(a + ma')x + (b + mb')y + c + mc' + d_t^2(x + my) = 0,$$

$$\text{or} \quad d_t^2(x + my + c_1) + (a + ma')(x + my + c_1) = 0 \quad (1),$$

$$\text{if} \quad \frac{b + mb'}{a + ma'} = m \quad (2), \qquad \frac{c + mc'}{a + ma'} = c_1;$$

therefore, integrating equation (1), and substituting successively the two values of m given by equation (2), we obtain the two required primitives; or if the values of m be equal, we must proceed as in the former case.

Ex. 1. $d_t^2 x - (3x + 4y - 3) = 0, \quad d_t^2 y + (x + y + 5) = 0.$

$$\therefore \quad d_t^2 (x + my) + (m - 3)\left(x + \frac{m-4}{m-3} y + \frac{3+5m}{m-3}\right) = 0.$$

Let $\dfrac{m-4}{m-3} = m$, or $m^2 - 4m + 4 = (m-2)^2 = 0$;

$$\therefore \quad d_t^2 (x + 2y) - (x + 2y - 13) = 0;$$

$$\therefore \quad x + 2y - 13 = ce^t + c'e^{-t},$$

and eliminating x from the latter of the given equations, we find

$$d_t^2 y - y + 18 + ce^t + c'e^{-t} = 0;$$

$$\therefore \quad y = 18 - \frac{c}{2}(t - \tfrac{1}{2})\, e^t + \frac{c'}{2}\, te^{-t} + ae^t + a'e^{-t},$$

and $x = -23 + c\,(t + \tfrac{1}{2})\, e^t - c'(t - 1)\, e^{-t} - 2ae^t - 2a'e^{-t}.$

Ex. 2. $d_t^2 x + ad_t x = 0, \quad d_t^2 y + ad_t y + b = 0.$

68. If we have three variables x, y, z, which are functions of t, and if

$$d_t x + ax + by + cz = T,$$

$$d_t y + a_1 x + b_1 y + c_1 z = T_1,$$

$$d_t z + a_2 x + b_2 y + c_2 z = T_2,$$

then if we multiply the second and third by indeterminate constants m, m', and add them to the first, and assume

$$b + b_1 m + b_2 m' = m\,(a + a_1 m + a_2 m'),$$

$$c + c_1 m + c_2 m' = m'\,(a + a_1 m + a_2 m'),$$

we have

$$d_t (x + my + m'z) + (a + ma_1 + m'a_2)(x + my + m'z)$$
$$= T + mT_1 + m'T_2.$$

To determine m, m', we obtain cubic equations; hence if m_1, m_2, m_3, m_1', m_2', m_3' be their roots, we have, by solving the linear equation,

$$x + m_1 y + m_1' z = F_1(t),$$

$$x + m_2 y + m_2' z = F_2(t),$$

$$x + m_3 y + m_3' z = F_3(t),$$

from which equations x, y, z may be found in terms of t.

Approximate Solutions of Differential Equations.

69. When all the known methods of integrating a proposed differential equation fail, we must endeavour to resolve it approximately, that is, to obtain from it the value of y in terms of x, in the form of a series. The first mode which presents itself of effecting this, is to assume for y a series arranged according to powers of x, with both its coefficients and exponents undetermined; for in most cases it happens that the exponents do not follow the progression of the natural numbers, and that particular artifices are requisite for discovering their law. When the form of this series is known, we determine its coefficients by substituting it and its differential coefficients, for y, $d_x y$, &c. in the proposed equation. The following application of the method to Riccati's Equation will give an idea of the mode of obtaining both the exponents and the coefficients.

Ex. $d_x y + by^2 - ax^n = 0$,

let $by = \dfrac{d_x z}{z}$; $\therefore\ d_x^2 z - abz x^n = 0$, or putting c for $-ab$,

$$d_x^2 z + czx^n = 0.$$

Assume $z = x^\alpha (A + Bx^\beta + Cx^{2\beta} + \&c.)$;

$$\left.\begin{aligned}\therefore\ &\alpha(\alpha-1)Ax^{\alpha-2}\\ &+(\alpha+\beta)(\alpha+\beta-1)Bx^{\alpha+\beta-2}+(\alpha+2\beta)(\alpha+2\beta-1)Cx^{\alpha+2\beta-2}+\&c.\\ &+\qquad cAx^{\alpha+n}\quad+\qquad\qquad cBx^{\alpha+\beta+n}+\&c.\end{aligned}\right\} = 0.$$

Hence β must equal $n+2$; and then to determine a, A, B, &c. we have

$$a(a-1)A = 0,$$

$$(a+n+2)(a+n+1)B + cA = 0,$$

$$(a+2n+4)(a+2n+3)C + cB = 0,$$

$$(a+3n+6)(a+3n+5)D + cC = 0,$$

....

Hence a may be either zero or unity, and A remains undetermined; calling therefore A and A' the two values of A corresponding to $a=0$, $a=1$, we get

$$z = A\left\{1 - \frac{cx^{n+2}}{(n+1)(n+2)} + \frac{c^2 x^{2n+4}}{(n+1)(n+2)(2n+3)(2n+4)} - \&c.\right\}$$

$$+A'x\left\{1 - \frac{cx^{n+2}}{(n+2)(n+3)} + \frac{c^2 x^{2n+4}}{(n+2)(n+3)(2n+4)(2n+5)} - \&c.\right\};$$

and substituting this value of z in the expression $y = \frac{d_x z}{bz}$, we shall obtain the value of y, involving only one arbitrary constant $\frac{A}{A'}$.

As the terms of the above series have divisors of the forms $(n+2)i \mp 1$, where i is an integer; if n be such that $(n+2)i \mp 1 = 0$, one or other of the series will be illusory, and we shall only obtain a particular value of z; and if $n+2=0$, both series become infinite, but in that case the equation may be exactly integrated by Art. 16.

70. In the preceding instance we arrive immediately at a complete result; but it often happens that the solution we obtain by the method of indeterminate coefficients involves no arbitrary constant. To supply this defect, we must introduce, instead of the arbitrary constant, a value of y cor-

responding to a given value of x; that is, supposing these to be b and a, we must substitute in the given equation,

$$y = b + u, \quad x = a + t;$$

then determine u in a series all whose terms vanish when $t = 0$; and replace u and t by their values $y - b$ and $x - a$; in this way it is evident that the arbitrary constant will be involved implicitly; for, from the complete integral

$$f(x, y, C) = 0,$$

C may be expressed in terms of a and b.

Ex. $\quad d_x y + y = gx^m,$

this becomes $\quad d_x u + b + u = g(a + t)^m;$

assume $\quad u = t^\alpha (A + Bt^\beta + Ct^\gamma + \&c.);$

$$\begin{aligned} \therefore \quad 0 = {} & \alpha A t^{\alpha-1} + (\alpha+\beta) B t^{\alpha+\beta-1} + (\alpha+\gamma) C t^{\alpha+\gamma-1} + \&c. \\ & + b \quad + \quad A t^\alpha \quad + \quad B t^{\alpha+\beta} \quad + \&c. \\ & - ga^m \quad - \quad mga^{m-1}t \quad - \frac{m(m-1)}{1.2} ga^{m-2}t^2 - \&c. \end{aligned}$$

$$\therefore \quad \alpha = 1, \quad \beta = 1, \quad \gamma = 2, \quad \&c.$$

$$A = ga^m - b, \quad 2B = gma^{m-1} - ga^m + b,$$

$$6C = gm(m-1)a^{m-2} - gma^{m-1} + ga^m - b, \quad \&c.$$

71. The approximate solution of a differential equation may sometimes be obtained in the form of a continued fraction by assuming

$$y = \frac{Ax^\alpha}{1 +} \, \frac{Bx^\beta}{1 +} \, \frac{Cx^\gamma}{1 + \&c.}.$$

First, suppose x to be very small, and for y substitute Ax^α in the given equation; then, retaining only the terms of

lowest dimensions in x, A and a become known by equating coefficients and exponents. Next, write $y=\dfrac{Ax^a}{1+z}$ in the proposed equation, and in the result put $z=Bx^\beta$, and determine B, β, as before, by supposing x to be very small; then in the transformed equation in z, put $z=\dfrac{Bx^\beta}{1+t}$; and so on for the rest.

Ex. $$my+(1+x)\,d_x y=0.$$

Let $y=Ax^a$; $\therefore\ (m+a)Ax^a+Aax^{a-1}=0$,

$$\text{or } Aax^{a-1}=0; \quad \therefore\ a=0,$$

and A remains undetermined. Next, put

$$y=\frac{A}{1+z}, \quad \text{and} \quad z=Bx^\beta,$$

and we get successively

$$m(1+z)=(1+x)\,d_x z;$$

$$m+Bx^\beta(m-\beta)=\beta Bx^{\beta-1}, \quad \text{or} \quad m=B\beta x^{\beta-1};$$

$$\therefore\ \beta=1, \quad B=m;$$

similarly, putting

$$z=\frac{mx}{1+t}, \quad \text{and} \quad t=Cx^\gamma,$$

we determine C and γ; and thus we obtain

$$y=\frac{A}{1+}\,\frac{mx}{1-}\,\frac{\frac{m-1}{1}\cdot\frac{x}{2}}{1+}\,\frac{\frac{m+1}{3}\cdot\frac{x}{2}}{1+}\,\frac{\frac{m-2}{3}\cdot\frac{x}{2}}{1-}\,\frac{\frac{m+2}{5}\cdot\frac{x}{2}}{1-}\,\frac{\frac{m-3}{5}\cdot\frac{x}{2}}{1+\&c.}.$$

Since the proposed equation when integrated gives

$$y = A(1+x)^{-m},$$

the above continued fraction is the development of $A(1+x)^{-m}$.

72. We may approximate to the integral of a differential equation by successive substitutions, in a manner similar to that invented by Newton for the solution of algebraical equations, as in the following instance.

Ex. $d_x^2 y + n^2 y + ay^2 + a = 0$, where a is a very small quantity,

we may assume

$$y = u + au_1 + a^2 u_2 + \&c.,$$

which gives

$$d_x^2 u + n^2 u + a + a(d_x^2 u_1 + n^2 u_1 + u^2) + a^2(d_x^2 u_2 + n^2 u_2 + 2uu_1)$$
$$+ a^3(d_x^2 u_3 + n^2 u_3 + 2uu_2) + \&c. = 0.$$

Hence, equating the coefficient of each power of a to zero, we get

$$d_x^2 u + n^2 u + a = 0,$$

$$d_x^2 u_1 + n^2 u_1 + u^2 = 0,$$

$$d_x^2 u_2 + n^2 u_2 + 2uu_1 = 0, \ \&c. \quad (3.)$$

The first give $u = -\dfrac{a}{n^2} + c\cos nx + c^1 \sin nx$;

and this value substituted in the second reduces it to the form

$$d_x^2 u_1 + n^2 u_1 = X_1,$$

the integral of which by Art. (58.) is

$$u_1 = \cos nx\left(c_1 - \frac{1}{n}\int_x X_1 \sin nx\right) + \sin nx\left(c_1' + \frac{1}{n}\int_x X_1 \cos nx\right).$$

Similarly, these values of u and u_1 substituted in (3) reduce it to the form

$$d_x^2 u_2 + n^2 u_2 = X_2,$$

which may be in like manner integrated; and in this way the coefficients of the powers of a may be deduced one from the other by a uniform process.

73. We have seen (Art. 57.) that the solution of

$$d_x^2 y + n^2 y = A \cos(mx + a) + B \cos(nx + \beta),$$

is $y = c_1 \cos nx + c_2 \sin nx$

$$+ \frac{A}{n^2 - m^2} \cos(mx + a) + \frac{B}{2n} x \sin(nx + \beta).$$

Hence, if from the proposed equation we had to determine y approximately, we could not neglect the term

$$A \cos(mx + a)$$

even when A is exceedingly small, provided m and n are nearly equal to one another; because in the value of y this term is divided by $n^2 - m^2$ which is very small. With respect to the last term in the value of y, we remark, that it is not periodical, but may increase indefinitely, as x increases. The equations which present themselves for solution in physical Astronomy are usually of the above form; and upon the peculiarities just noticed depend some of the most interesting results in that subject.

SECTION V.

ON DIFFERENTIAL EQUATIONS INVOLVING TWO OR MORE INDEPENDENT VARIABLES.

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74. In this part of the subject, it is convenient to employ a system of notation which we have not hitherto found it necessary to notice, and which may be explained as follows.

If u be a function of any number of independent variables x, y, z, t, &c., and if $u + \delta u$ be the value of u when these variables simultaneously become

$$x + \delta x, \quad y + \delta y, \quad z + \delta z, \quad \&c.,$$

then $\delta u = d_x u \,.\, \delta x + d_y u \,.\, \delta y + d_z u \,.\, \delta z + \dots$

$$+ \text{ terms of two dimensions in } \delta x, \delta y, \&c.$$

If we now agree to neglect all terms of higher dimensions than one in δx, δy, &c., and denote the value of δu corresponding to that supposition by du, we have

$$du = d_x u \,.\, \delta x + d_y u \,.\, \delta y + d_z u \,.\, \delta z + \dots;$$

according to which definition, it appears that du denotes that part of the increment of u, as given by Taylor's Theorem, which involves only the first powers of the arbitrary increments of the variables on which it depends; hence, in conformity with this definition, δx, δy, δz, &c., must be represented by dx, dy, dz, &c.; for if $f(x) = x$, then

$$f(x + \delta x) = x + \delta x;$$

consequently, that part of the increment of $f(x)$ which involves the first power of δx is, in this case, the whole of it, δx, therefore, $dx = \delta x$; and so on for the others;

$$\therefore \; du = d_x u \,.\, dx + d_y u \,.\, dy + d_z u \,.\, dz + \&c.,$$

where dx, dy, dz, &c., are the arbitrary increments of the independent variables x, y, z, &c., and are entirely independent of those variables; and du is that part of the corresponding increment of the dependent variable u, which involves only simple dimensions of dx, dy, dz, &c.; and which approaches nearer and nearer to the value of the whole increment of u, the smaller dx, dy, dz, &c. are taken.

The quantities $d_x u \,.\, dx$, $d_y u \,.\, dy$, &c., are called the partial differentials of u with respect to x, y, &c. respectively; and du is called the total or complete differential of u, or the differential of u, merely.

75. According to the above definition, the differential of du, or the second differential of u, will be, supposing it to involve only two independent variables,

$$\begin{aligned} d^2 u &= d_x (du)\, dx + d_y (du)\, dy \\ &= d_x (d_x u \,.\, dx + d_y u \,.\, dy)\, dx + d_y (d_x u \,.\, dx + d_y u \,.\, dy)\, dy \\ &= d_x^2 u \,.\, dx^2 + 2 d_x d_y u \,.\, dx\, dy + d_y^2 u \,.\, dy^2, \end{aligned}$$

because in this process dx and dy are entirely independent of x and y; being, in fact, the arbitrary increments of those variables, which might have been denoted by h and k, did not a due regard to the precision and symmetry of the notation require it otherwise. And in general, if u be a function of any number of independent variables x, y, z, t, &c., the n^{th} differential of u will be given by the formula, (where the symbols of operation are separated from those of quantity,)

$$d^n u = (dx \,.\, d_x + dy \,.\, d_y + dz \,.\, d_z + \&c.)^n u.$$

76. Hence, we see that the differential of a function has no reference to one variable rather than to another; but in its formation, all the variables of which it is a function are supposed to undergo simultaneous unconnected alterations; and the value of the differential depends upon the values of all those alterations. Whereas in forming a differential coefficient, one particular independent variable only is changed, the rest remaining unaffected; and a quantity is produced, wholly independent of the value of the alteration which that variable may have received.

77. It is evident that the rules for deriving differential coefficients, suffice for finding the differentials of functions.

Having, for instance, formed all the partial differential coefficients of the first order, if we multiply each by the arbitrary increment (or differential) of the corresponding independent variable, and take the sum, we shall obtain the first differential of the function; and similarly, the differentials of the second, and higher orders, may be formed by means of the formula at the end of Article 75.

78. Total integration of a proposed differential, is the finding a function whose differential is the quantity proposed. This operation is denoted by the symbol $\int$ without any affixed variable; this process, like the former of differtiation, having reference to all, and not to one particular variable.

Ex. If $u = x^2 + y^2$,

$$du = 2(xdx + ydy) \text{ is the differential of } u;$$

also $\int(xdx + ydy) = \frac{u}{2}$, since the latter quantity differentiated produces the quantity under the symbol of total integration.

When a quantity is presented for total integration, it must be reducible to the form $f(v)dv$; and then its integral $= \int_v f(v)$. After this observation, the rules of ordinary in-

tegration will enable us to perform the process of total integration.

Ex. To integrate $\dfrac{y(xdy - ydx)}{x^2\sqrt{x^2+y^2}}$,

$$\int \frac{y(xdy - ydx)}{x^2\sqrt{x^2+y^2}} = \int \frac{\frac{y}{x}}{\sqrt{1+\left(\frac{y}{x}\right)^2}} \cdot \frac{xdy - ydx}{x^2}$$

$$= \int \frac{v}{\sqrt{1+v^2}}\, dv, \text{ putting } \frac{y}{x} = v;$$

$$= \int_v \frac{v}{\sqrt{1+v^2}} = \sqrt{1+v^2} + C$$

$$= \frac{1}{x}\sqrt{x^2+y^2} + C.$$

79. Differential Equations involving more than two variables admit of division into two classes, Total and Partial.

A total differential equation is one which expresses the differential of the dependent variable in terms of the other variables and their differentials, and sometimes also of the dependent variable itself; it is consequently equivalent to a system of equations in which each differential coefficient of the dependent variable is given explicitly. Thus, u being a function of the independent variables x, y, z, the total differential equation

$$du = Pdx + Qdy + Rdz$$

amounts to the same as the system of equations,

$$d_x u = P, \quad d_y u = Q, \quad d_z u = R.$$

Also, z being a function of the independent variables x and y, the total differential equation

$$Pdx + Qdy + Rdz = 0$$

amounts to the same as the two equations,

$$P + Rd_x z = 0, \quad Q + Rd_y z = 0.$$

A partial differential equation, on the contrary, is only a relation between all or certain of the partial differential coefficients of the dependent variable, and the variables; as in the instances,

$$(x - a)\,d_x z + (y - b)\,d_y z = z - c,$$

$$x^2 d_x^2 z + 2xy\,d_x d_y z + y^2 d_y^2 z = 0,$$

z in both being a function of the independent variables x and y.

Total Differential Equations.

80. To integrate $du = Pdx + Qdy$, u being a function of the independent variables x and y, and P and Q being functions of x and y.

Since $Pdx + Qdy$ must be identical with $d_x u\,dx + d_y u\,dy$, we have

$$d_x u = P, \quad d_y u = Q, \text{ with the condition } d_y P = d_x Q;$$

$$\therefore \quad u = \int_x P + f(y), \quad f(y) \text{ involving } y \text{ only};$$

$$\therefore \quad Q = d_y u = d_y\left(\int_x P\right) + d_y f(y);$$

$$\therefore \quad f(y) = \int_y \left\{Q - d_y\left(\int_x P\right)\right\};$$

consequently u is known.

Ex. $$du = \frac{y}{(x+y)^2}\,dx - \frac{x}{(x+y)^2}\,dy,$$

$$d_x u = \frac{y}{(x+y)^2}, \quad d_y u = -\frac{x}{(x+y)^2},$$

$$\therefore \quad u = -\frac{y}{x+y} + f(y)$$

$$-\frac{x}{(x+y)^2} = -\frac{1}{x+y} + \frac{y}{(x+y)^2} + d_y f(y);$$

$$\therefore \quad d_y f(y) = 0, \qquad \therefore \quad f(y) = C,$$

$$\therefore \quad u = C - \frac{y}{x+y}.$$

81. To integrate $du = Pdx + Qdy + Rdz$, u being a function of the independent variables x, y, z.

Since $Pdx + Qdy + Rdz$ is identical with

$$d_x u \,.\, dx + d_y u \,.\, dy + d_z u \,.\, dz,$$

we have $d_x u = P, \quad d_y u = Q, \quad d_z u = R;$

together with the equations of condition for being exact

$$d_y P = d_x Q, \quad d_z Q = d_y R, \quad d_x R = d_z P,$$

which are found by supposing each of the quantities z, x, y to be constant in succession (as we are evidently at liberty to do, since those variables are independent of one another), and then taking the corresponding equation of condition for du being the exact differential of a function of two independent variables. Hence

$$u = \textstyle\int_x P + w, \quad w \text{ being a function of } y \text{ and } z;$$

also $Q = d_y (\int_x P) + d_y w,$

$R = d_z (\int_x P) + d_z w,$

which two equations giving the values of the partial differential coefficients of w, its value may be found by the preceding Article; and so the value of u completely determined.

Ex. 1. $du = yzdx + xzdy + xydz$;

$$d_x u = yz, \quad d_y u = xz, \quad d_z u = xy,$$

$$\therefore \ u = xyz + w;$$

$$xz = xz + d_y w, \qquad \text{or} \quad d_y w = 0,$$

$$xy = xy + d_z w, \qquad \text{or} \quad d_z w = 0,$$

$$\therefore \ w = C;$$

$$\therefore \ u = xyz + C.$$

Ex. 2. $du = \dfrac{y}{a-z}dx + \dfrac{x}{a-z}dy + \dfrac{xy}{(a-z)^2}dz$;

$$d_x u = \frac{y}{a-z}, \quad d_y u = \frac{x}{a-z}, \quad d_z u = \frac{xy}{(a-z)^2};$$

$$u = \frac{xy}{a-z} + C.$$

Ex. 3. $du = a(xdx + ydy) + bdz$;

$$u = \frac{a}{2}(x^2 + y^2) + bz + C.$$

82. We next come to the consideration of the equation

$$Pdx + Qdy + Rdz = 0,$$

one of the three variables x, y, z, being a function of the other two, which are independent.

Since the proposed equation may arise from combining the results of differentiating two separate equations, we have first to examine whether it can be satisfied by a single primitive relation between x, y and z. If x, y, z be co-ordinates of a point, the cases will be distinguished according as the proposed equation is the differential equation to a series of surfaces, or a series of curves.

For example, the equation

$$(z-c)\,x\,dx + (z-c)\,y\,dy = \{x(x-a) + y(y-b)\}\,dz,$$

arises from combining the results of differentiating the equations

$$z = f(x^2 + y^2), \quad \frac{y-b}{z-c} = \phi\left(\frac{x-a}{z-c}\right);$$

for these equations give respectively

$$y d_x z - x d_y z = 0, \quad (x-a)\,d_x z + (y-b)\,d_y z = z - c;$$

from which if $d_x z$ and $d_y z$ be determined, and substituted in

$$dz = d_x z \,.\, dx + d_y z \,.\, dy,$$

we get the proposed equation; which, consequently, cannot in general be satisfied by a single relation between x, y, z. It is the analytical expression of the conditions of the problem, to find a surface belonging at the same time to conical surfaces, and surfaces of revolution about the axis of z.

83. To find the equation of condition for

$$P dx + Q dy + R dz = 0,$$

admitting a solution of the form $f(x, y, z) = C$.

If the proposed equation can be satisfied by a single relation between x, y, and z,

$$dz = -\frac{P}{R}\,dx - \frac{Q}{R}\,dy,$$

is the differential of a function of two independent variables;

$$\therefore \; d_x z = -\frac{P}{R}, \quad d_y z = -\frac{Q}{R},$$

with the condition $d_y\left(\frac{P}{R}\right) = d_x\left(\frac{Q}{R}\right)$;

or, since P, Q, R may contain z, which is a function of x and y,

$$d_{(y)}\left(\frac{P}{R}\right) + d_{(z)}\left(\frac{P}{R}\right) d_y z = d_{(x)}\left(\frac{Q}{R}\right) + d_{(z)}\left(\frac{Q}{R}\right) d_x z;$$

therefore, substituting for $d_x z$, $d_y z$, their values,

$$R d_{(y)}\left(\frac{P}{R}\right) - Q d_{(z)}\left(\frac{P}{R}\right) = R d_{(x)}\left(\frac{Q}{R}\right) - P d_{(z)}\left(\frac{Q}{R}\right);$$

and performing the differentiations, and reducing, we get

$$P\{d_{(y)}R - d_{(z)}Q\} + Q\{d_{(z)}P - d_{(x)}R\} + R\{d_{(x)}Q - d_{(y)}P\} = 0,$$

the equation of condition for

$$Pdx + Qdy + Rdz = 0;$$

admitting a solution of the form $f(x, y, z) = C$.

84. If the above equation of condition is not satisfied, then the equation

$$Pdx + Qdy + Rdz = 0$$

cannot, by being multiplied by any factor, become susceptible of a solution of the form

$$f(x, y, z, C) = 0.$$

For suppose V to be a factor which renders

$$Pdx + Qdy + Rdz$$

the immediate differential of some function u of x, y, z, considered as independent;

$$\therefore\ d_{(y)}(VP) = d_{(x)}(VQ), \qquad d_{(z)}(VQ) = d_{(y)}(VR),$$
$$d_{(x)}(VR) = d_{(z)}(VP);$$

$$\text{or}\quad \left.\begin{aligned} Vd_{(y)}P + Pd_{(y)}V &= Vd_{(x)}Q + Qd_{(x)}V \\ Vd_{(z)}Q + Qd_{(z)}V &= Vd_{(y)}R + Rd_{(y)}V \\ Vd_{(x)}R + Rd_{(x)}V &= Vd_{(z)}P + Pd_{(z)}V \end{aligned}\right\} \quad (1).$$

Hence, multiplying the first of these equations by R, the second by P, and the third by Q, and taking their sum, the factor V disappears, and we find

$$P(d_{(y)}R - d_{(z)}Q) + Q(d_{(z)}P - d_{(x)}R) + R(d_{(x)}Q - d_{(y)}P) = 0, \quad (2)$$

the same equation of condition as in the preceding Article.

85. If this equation be satisfied, which will be the case only when the proposed admits a primitive of the form

$$f(x, y, z, C) = 0,$$

equations (1) afford a means of determining V. Then

$$du = VPdx + VQdy + VRdz,$$

whence u can be found by the method of Art. 81, and $u + C = 0$ is the required relation between x, y, and z.

Or, without determining V, we may integrate considering one of the variables constant, and add an arbitrary function of that variable; then differentiate the result with respect to that variable and compare it with the proposed equation, and so the correction will become known.

Obs. In the majority of cases which present themselves, the factor V is capable of being determined by inspection.

Ex. $(ay - bz)\,dx + (cz - ax)\,dy + (bx - cy)\,dz = 0.$

Divide by $(ay - bz)(bx - cy)$, and the resulting equation

$$\frac{dx}{bx - cy} + \frac{(cz - ax)\,dy}{(ay - bz)(bx - cy)} + \frac{dz}{ay - bz} = 0,$$

satisfies the conditions, considering x, y, z, as independent,

$$d_y P = d_x Q, \quad d_z Q = d_y R, \quad d_x R = d_z P;$$

and therefore the first member may be regarded as the differential of some function, u, of x, y, z considered as independent;

$$\therefore\ d_x u = \frac{1}{bx - cy};$$

$$\therefore\ u = \frac{1}{b} \log (bx - cy) + w;$$

$$\therefore \quad \frac{1}{ay - bz} = d_z w,$$

$$\frac{cz - ax}{(ay - bz)(bx - cy)} = -\frac{c}{b} \cdot \frac{1}{bx - cy} + d_y w,$$

$$\text{or } d_y w = -\frac{a}{b} \cdot \frac{1}{ay - bz};$$

$$\therefore \quad dw = -\frac{1}{b} \cdot \frac{ady - bdz}{ay - bz};$$

$$\therefore \quad w = -\frac{1}{b} \log (ay - bz) + C;$$

$$\therefore \quad u = \frac{1}{b} \log \left(\frac{bx - cy}{ay - bz}\right) + C = 0, \quad \text{or } \frac{bx - cy}{ay - bz} = C.$$

Obs. When in an equation of this sort, the differentials enter above the first degree, it is not integrable unless it can be resolved into rational factors of the form $Pdx + Qdy + Rdz$; for whatever be the integral, it must upon differentiation produce a result of that form.

86. If the equation

$$Pdx + Qdy + Rdz = 0$$

be susceptible of a primitive of the form $f(x, y, z, C) = 0$, and be homogeneous and of n dimensions with respect to x, y, z; then, putting $x = vz$, $y = wz$, and dividing by z^n, it becomes

$$S(vdz + zdv) + T(wdz + zdw) + Udz = 0,$$

S, T, U being functions of v and w;

$$\text{or } \quad -\frac{dz}{z} = \frac{Sdv + Tdw}{Sv + Tw + U};$$

hence the second member is an exact differential since the first is so, and it may be generally integrated by inspection, or by the method of Art. 80.

Ex. 1. $$(y + z)\,dx + (x + z)\,dy + (x + y)\,dz = 0;$$

put $x = vz, \quad y = wz,$

$$\therefore \frac{dz}{z} + \frac{(w + 1)\,dv + (v + 1)\,dw}{v(w + 1) + w(v + 1) + v + w} = 0,$$

$$\text{or} \quad \frac{dz}{z} + \tfrac{1}{2}\,\frac{d(v + w + vw)}{v + w + vw} = 0;$$

$$\therefore \log z^2(v + w + vw) = \log C, \quad \text{or} \quad xz + yz + xy = C.$$

Ex. 2. $$(y^2 + yz + z^2)\,dx + (x^2 + xz + z^2)\,dy$$
$$+ (x^2 + xy + y^2)\,dz = 0.$$

Make $x = vz, \quad y = wz,$

$$\therefore \frac{dz}{z} + \frac{(w^2 + w + 1)\,dv + (v^2 + v + 1)\,dw}{v(w^2 + w + 1) + w(v^2 + v + 1) + v^2 + vw + w^2} = 0,$$

but of the latter fraction, the denominator

$$= vw(w + v + 1) + v(w + 1 + v) + w(v + 1 + w)$$
$$= (vw + v + w)(v + w + 1),$$

and the numerator

$$= (1 + v + w)\,d(v + w + vw) - (v + w + vw)\,d(v + w);$$

$$\therefore \frac{dz}{z} + \frac{d(v + w + vw)}{v + w + vw} - \frac{d(1 + v + w)}{1 + v + w} = 0.$$

$$\therefore \log\left\{\frac{z(v + w + vw)}{1 + v + w}\right\} = \log C;$$

$$\therefore zx + zy + xy = C(z + x + y).$$

Ex. 3. $$(x^2 - y^2 + z^2)\,dx - z^2 dy$$
$$+ (y - x)(z^2 + xy + x^2)\,\frac{dz}{z} = 0,$$

putting $x = vz$, $y = wz$, this is reduced to

$$dw = (1 + v^2 - w^2)\,dv;$$

$$\therefore\ \frac{e^{-v^2}}{w - v} = \int_v e^{-v^2} + C, \quad \text{(Ex. 3. Art. 18.)}$$

$$\text{or}\ \frac{z^2 e^{-\frac{x^2}{z^2}}}{y - x} = \int_x e^{-\frac{x^2}{z^2}} + Cz,$$

z being constant under the sign of integration.

Total Differential Equations that do not admit of a Single Primitive.

87. We have seen that the equation

$$Pdx + Qdy + Rdz = 0,$$

when the equation of condition (Art. 83) is not satisfied, does not admit of being derived from a single primitive equation involving two independent variables. The integral in this case will be exhibited by a system of two equations; and the proposed equation cannot be regarded as the differential equation to a surface, but to a system of curves in space, all endowed with some common property.

Ex. 1. $dz = aydx + bdy$.

Since the equation of condition in this case is not satisfied, x and y cannot be independent, and we may assume $y = f(x)$;

$$\therefore\ dz = af(x)\,dx + bf'(x)\,dx;$$

$$\therefore\ z = a\int_x f(x) + bf(x),$$

which, with $y = f(x)$, constitutes the integral of the proposed equation.

In general, if V be a factor which makes $Pdx + Qdy$ an exact differential, considering z as constant, and we find

$$\int (VPdx + VQdy) = w + \phi(z);$$

it is evident that

$$w + \phi(z) = 0, \text{ together with } d_z w + \phi'(z) - VR = 0,$$

satisfy the proposed equation, where $\phi(z)$ denotes any function of z

Ex. 2. $z dx + x dy + y dz = 0$,

$$y + z \log x + \phi(z) = 0, \quad \log x + d_z \phi(z) = \frac{y}{x}.$$

Ex. 3. $\{x(x-a) + y(y-b)\} dz = (z-c)(x dx + y dy)$,

$$x^2 + y^2 + 2\phi(z) = 0, \quad x(x-a) + y(y-b) + (z-c)\phi'(z) = 0.$$

Partial Differential Equations.

88. In partial differential equations of two independent variables, the differential coefficients of the first order $d_x z$, $d_y z$, of the dependent variable z, are usually denoted by the symbols p and q; and $d_x^2 z$, $d_x d_y z$, $d_y^2 z$, the differential coefficients of the second order, by r, s, t, respectively. A partial differential equation is said to be of the n^{th} order, when it involves one or more of the partial differential coefficients of the dependent variable of the n^{th} order; but none of a superior order. To be the general equation of the n^{th} order, it ought to contain the independent variables, and the dependent variable together with all its partial differential coefficients from the first order to the n^{th} order inclusive. To integrate a partial differential equation, is to find for the dependent variable, an expression between the differential coefficients of which, that relation exists which is indicated by the proposed equation; and under the most general form possible.

Equations of the First Order.

89. The complete integral of $f(x, y, z, p, q) = 0$, the general equation of the first order, will involve one arbitrary or general function.

For let $u = F\{x, y, z, \phi(v)\} = 0$, be an equation by virtue of which z is a function of the independent variables x and y, v being a known function of x, y, and z. Then p and q are given by the equations $d_x u = 0$, $d_y u = 0$, each of which will involve $d_v \phi(v)$ or $\phi'(v)$, and may involve $\phi(v)$; consequently, between the three equations $u = 0$, $d_x u = 0$, $d_y u = 0$, it will be possible to eliminate $\phi(v)$ and $\phi'(v)$, and there will result a relation $f(x, y, z, p, q) = 0$, wholly independent of the form of the function $\phi(v)$; and it is evident that in general more than one function cannot thus be eliminated. Conversely, an equation of the form

$$f(x, y, z, p, q) = 0$$

being proposed, its complete integral to have all the generality possible, must be of the form $F\{x, y, z, \phi(v)\} = 0$, where the form of $\phi(v)$ is perfectly arbitrary.

For example, let

$$u = z + mx + ny - \phi(v) = 0,$$

$$\text{where} \quad v = (x-a)^2 + (y-b)^2 + (z-c)^2;$$

$$\text{then} \quad d_x u = p + m - \phi'(v)\{2(x-a) + 2(z-c)p\} = 0,$$

$$d_y u = q + n - \phi'(v)\{2(y-b) + 2(z-c)q\} = 0.$$

Hence, transposing, and dividing one result by the other to eliminate $\phi'(v)$, we find

$$\frac{p+m}{q+n} = \frac{(x-a) + (z-c)p}{(y-b) + (z-c)q},$$

$$\text{or} \quad p\{y - b - n(z-c)\} - q\{x - a - m(z-c)\}$$
$$= n(x-a) - m(y-b),$$

the partial differential equation of which the complete integral is

$$z + mx + ny = \phi\{(x-a)^2 + (y-b)^2 + (z-c)^2\}.$$

90. To integrate an equation in which only one of the differential coefficients of the first order enters with x, y, and z.

Let the equation be

$$f(x, y, z, p) = 0.$$

Integrate it, considering y as a constant, and in place of the arbitrary constant C, add a function of y of arbitrary form. The resulting solution, containing one arbitrary function, will have all the generality that can be attained.

Ex. $$d_x z\,(x^2 + y^2) = z^2 + y^2;$$

$$\therefore\ \frac{d_x z}{z^2 + y^2} - \frac{1}{x^2 + y^2} = 0;$$

$$\therefore\ \tan^{-1}\frac{z}{y} - \tan^{-1}\frac{x}{y} = \tan^{-1}\phi(y),$$

$$\text{or}\quad z - x = (y^2 + xz)\,\phi(y),$$

$\phi(y)$ being arbitrary in form.

The equation $f(x, y, z, q) = 0$ is similarly integrated, the correction in this case being an arbitrary function of x.

Ex. $$xy\,d_y z + nz = 0,$$

$$z^x y^n = \phi(x).$$

91. To integrate the linear equation of the first order,

$$Pp + Qq = R,$$

P, Q, and R being functions of x, y, and z.

Let the primitive be $F(x, y, z) = 0$; therefore, denoting $d_{(x)}F(x, y, z)$ by $F'(x)$, and so on for the other coefficients, we get

$$F'(x) + F'(z)\,.\,p = 0,$$

$$F'(y) + F'(z)\,.\,q = 0;$$

$$\therefore \quad PF'(x) + QF'(y) + RF'(z) = 0.$$

But $dF(x, y, z) = F'(x)\,dx + F'(y)\,dy + F'(z)\,dz = 0$;

$$\therefore \quad PF'(x)\,dx + PF'(y)\,dy + PF'(z)\,dz = 0,$$

and $PF'(x)\,dx + QF'(y)\,dx + RF'(z)\,dx = 0$;

$$\therefore \quad F'(y)\{Pdy - Qdx\} + F'(z)\{Pdz - Rdx\} = 0, \quad (1).$$

which is satisfied by

$$\left.\begin{aligned} Pdy - Qdx &= 0, \\ Pdz - Rdx &= 0. \end{aligned}\right\} \quad (2).$$

Suppose that by integrating these equations, either separately or conjointly, we obtain $M = a$, $N = b$, two relations between the three variables and the arbitrary constants a and b, which satisfy them. By means of these two equations, any two of the variables as y and z can be expressed in terms of a and b and the third variable x. The complete primitive becomes

$$F(x, y, z) = \phi(x, a, b) = 0,$$

and the differential of $\phi(x, a, b)$ must by virtue of equations (2) be identically equal to zero, therefore $\phi(x, a, b)$ cannot contain x, and

$$0 = \phi(a, b) = \phi(M, N),$$

$$\text{or} \quad M = f(N).$$

The assumptions (2) satisfy equation (1) independently of the forms of $F'(y)$, $F'(z)$, that is, independently of the form of F; therefore the form of ϕ, and consequently of f, is arbitrary. Hence $M = f(N)$ being a solution of the proposed equation and containing one arbitrary function, is the general primitive.

92. For the success of the method, it is in general necessary that of the three equations

$$Pdy - Qdx = 0, \quad Pdz - Rdx = 0, \quad Qdz - Rdy = 0,$$

one at least should contain those two variables only whose increments or differentials appear in it. They are called the reducing equations of $Pp + Qq = R$; and may be easily remembered under the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

93. By a similar process, partial differential equations of three or more independent variables can sometimes be integrated. If z be a function of x, y, t, and

$$Tn + Pp + Qq = R,$$

where $n = d_t z$, we have

$$n = \frac{R - Pp - Qq}{T};$$

which, substituted in

$$dz = ndt + pdx + qdy,$$

gives $Tdz - Rdt = p(Tdx - Pdt) + q(Tdy - Qdt)$.

And if the equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R} = \dfrac{dt}{T}$, or

$$Tdz - Rdt = 0, \quad Tdx - Pdt = 0, \quad Tdy - Qdt = 0,$$

$$\text{give } L = a, \quad M = b, \quad N = c,$$

then the solution is

$$\phi(L, M, N) = 0, \quad \text{or} \quad L = f(M, N).$$

94. To explain the geometrical meaning of the solution of a partial differential equation of the first order.

If we regard x, y, z as the co-ordinates of a point, and the proposed differential equation as the equation to a system of surfaces, $M=a$, $N=b$, are the equations to two surfaces, which, being satisfied by values of x, y, z which satisfy the differential equation, conjointly represent a line on a surface represented by the primitive. By giving any alterations to a and b we obtain other lines in space; but in order that these lines may always lie on the surface in question, these alterations must not be independent but connected, according to some law expressed by the relation

$$a=f(b), \qquad \text{or } M=f(N).$$

For example, the equation

$$px+qy=z$$

is the differential equation to a conical surface, and

$$\frac{y}{z}=a, \qquad \frac{x}{z}=b,$$

are the equations to any straight line through the vertex. If a and b undergo all possible variations consistently with the restriction imposed by the nature of the directrix, the assemblage of straight lines thus formed is the conical surface.

We shall now apply the preceding method to examples.

Ex. 1. $$px+qz+y=0,$$

$$dz=pdx+qdy,$$

$$xdz=pxdx+qxdy$$

$$-ydx=pxdx+qzdx;$$

$$\therefore\ xdz+ydx=q(xdy-zdx);$$

$$\therefore\ xdz+ydx=0,$$

$$xdy-zdx=0,$$

$$zdz+ydy=0.$$

From the last, $z^2 + y^2 = a^2$;

$$\therefore\ xdz + \sqrt{a^2 - z^2}\,dx = 0,$$

$$\therefore\ \frac{dz}{\sqrt{a^2 - z^2}} + \frac{dx}{x} = 0;$$

$$\therefore\ \sin^{-1}\frac{z}{a} + \log x = \log b;$$

$$\therefore\ xe^{\tan^{-1}\frac{z}{y}} = f(\sqrt{y^2 + z^2}).$$

Ex. 2. $p(x - a) + q(y - b) = z - c.$

This compared with $Pp + Qq = R$, gives in the place of

$$Pdz - Rdx = 0, \quad Qdz - Rdy = 0,$$

the reducing equations

$$(x - a)\,dz - (z - c)\,dx = 0, \quad (y - b)\,dz - (z - c)\,dy = 0;$$

$$\therefore\ \frac{x - a}{z - c} = \alpha, \quad \frac{y - b}{z - c} = \beta;$$

$$\therefore\ \frac{y - b}{z - c} = f\left(\frac{x - a}{z - c}\right).$$

Ex. 3. $mp + nq = 1$;

$$\therefore\ mdz - dx = 0, \quad ndz - dy = 0;$$

$$\therefore\ x - mz = \alpha, \quad y - nz = \beta;$$

$$\therefore\ y - nz = f(x - mz).$$

Ex. 4. $(x - mz)\,p + (y - nz)\,q = 0$;

$$\therefore\ (x - mz)\,dz = 0; \quad \therefore\ z = \alpha,$$

$$(x - mz)\,dy - (y - nz)\,dx = 0,$$

$$\text{or}\quad (x - ma)\,dy - (y - na)\,dx = 0;$$

$$\therefore\ \frac{x - ma}{y - na} = b;\ \text{but } a = f(b),$$

$$\therefore\ z = f\left(\frac{x - mz}{y - nz}\right).$$

5. $px + qy = nz, \quad z = x^n f\left(\frac{y}{x}\right).$

6. $\frac{p}{y} + \frac{q}{x} = \frac{1}{z}, \quad z^2 = xy + f\left(\frac{y}{x}\right).$

7. $px^2 + qy^2 = z^2, \quad \frac{1}{z} = \frac{1}{x} + f\left(\frac{1}{y} - \frac{1}{x}\right).$

8. $x d_x z + y d_y z + t d_t z + u d_u z = nz, \quad z = t^n f\left(\frac{x}{t}, \frac{y}{t}, \frac{u}{t}\right).$

9. $z - px - qy = n\sqrt{x^2 + y^2 + z^2},$

$$x^{n-1}\left(z + \sqrt{x^2 + y^2 + z^2}\right) = f\left(\frac{y}{x}\right).$$

10. $(ax + by)\,p + (a'x + b'y)\,q = cz.$ Assume $z = e^u$.

95. The following examples require artifices in the combination of the reducing equations.

Ex. 1. $\{y - b - n(z - c)\}\,p - \{x - a - m(z - c)\}\,q$

$$= n(x - a) - m(y - b),$$

the equation to surfaces of revolution. Here the reducing equations are

$$\{n(x-a)-m(y-b)\}\,dx-\{y-b-n(z-c)\}\,dz=0, \quad (1).$$

$$\{y-b-n(z-c)\}\,dy+\{x-a-m(z-c)\}\,dx=0, \quad (2).$$

$$\{x-a-m(z-c)\}\,dz+\{n(x-a)-m(y-b)\}\,dy=0, \quad (3).$$

Multiply (1) by $x-a$, and (3) by $y-b$, and add,

$$\therefore\ \{n(x-a)-m(y-b)\}\times$$
$$\{(x-a)\,dx+(y-b)\,dy+(z-c)\,dz\}=0;$$

$$\therefore\ (x-a)\,dx+(y-b)\,dy+(z-c)\,dz=0;$$

$$\therefore\ (x-a)^2+(y-b)^2+(z-c)^2=\alpha.$$

Multiply (1) by m, and (3) by n, and add,

$$\therefore\ \{n(x-a)-m(y-b)\}\,(mdx+ndy+dz)=0;$$

$$\therefore\ mdx+ndy+dz=0,$$

$$mx+ny+z=\beta=f(\alpha);$$

$$\therefore\ z+mx+ny=f\{(x-a)^2+(y-b)^2+(z-c)^2\}.$$

Ex. 2. $(x+y+z)\,d_t z+(t+y+z)\,d_x z$

$$+(t+x+z)\,d_y z=t+x+y.$$

The reducing equations are

$$(x+y+z)\,dz=(t+x+y)\,dt,$$

$$(x+y+z)\,dx=(t+y+z)\,dt,$$

$$(x+y+z)\,dy=(t+x+z)\,dt;$$

$$\therefore\ (x+y+z)(dz-dt)=(t-z)\,dt,$$

$$(x+y+z)(dx-dt)=(t-x)\,dt,$$

$$(x+y+z)(dy-dt)=(t-y)\,dt.$$

Also,

$$(x+y+z)(dx+dy+dz+dt)=3(t+x+y+z)\,dt;$$

$$\therefore\ \frac{dz-dt}{t-z}=\frac{dt}{x+y+z}=\frac{dx+dy+dz+dt}{3(x+y+z+t)};$$

$$\therefore\ (x+y+z+t).(z-t)^3=\alpha,$$

Similarly, $(x+y+z+t).(x-t)^3=\beta,$

$$(x+y+z+t).(y-t)^3=\gamma;$$

$$\therefore\ (x+y+z+t).(z-t)^3$$

$$=F\left\{(x+y+z+t).(x-t)^3,\ (x+y+z+t).(y-t)^3\right\}.$$

96. To integrate $F(x, y, z, p, q)=0$, when it contains terms of more than one dimension in p and q.

In order that

$$dz=p\,dx+q\,dy$$

may be a perfect differential, we must have

$$d_{(y)}p+q\,d_{(z)}p=d_{(x)}q+p\,d_{(z)}q. \qquad (2).$$

If from the proposed equation we determine

$$q,\quad d_{(x)}q,\quad d_{(z)}q,$$

equation (2) becomes an equation for the determination of p of the form

$$d_{(x)}p+M\,d_{(y)}p+N\,d_{(z)}p=L,$$

the integration of which depends on the integration of one of the equations

$$dp - L\,dx = 0, \qquad dy - M\,dx = 0, \qquad dz - N\,dx = 0.$$

Let $p = f(x, y, z, a)$ be found from these equations, a being an arbitrary constant. This value of p and the corresponding value of q found from the proposed equation, being substituted in $dz = p\,dx + q\,dy$ render it an exact differential; and thus a value of z will be obtained involving two arbitrary constants a and b; and this will consequently be the complete primitive. The general primitive may be obtained by putting $b = \psi(a)$, differentiating the equation with regard to a, and eliminating a. The result containing one arbitrary function is as general as any solution which the equation admits.

Ex. 1. $p^2 + q^2 = 1$,

$$q = \sqrt{1 - p^2}, \quad d_x q = -\frac{p}{\sqrt{1-p^2}} d_x p, \quad d_z q = -\frac{p}{\sqrt{1-p^2}} d_z p;$$

$$\therefore\ d_y p + \sqrt{1-p^2}\, d_x p + \frac{p\,d_x p}{\sqrt{1-p^2}} + \frac{p^2 d_z p}{\sqrt{1-p^2}} = 0,$$

$$dp = 0,$$

$$\frac{p}{\sqrt{1-p^2}} dy - dx = 0,$$

$$\frac{p}{\sqrt{1-p^2}} dz - \frac{dx}{\sqrt{1-p^2}} = 0;$$

$$\therefore\ p = a, \quad q = \sqrt{1-a^2},$$

$$dz = a\,dx + \sqrt{1-a^2}\,dy,$$

$$z = ax + \sqrt{1-a^2}\,y + b,$$

$$\text{or}\quad z = ax + \sqrt{1 - a^2}\,y + \phi(a),$$

$$0 = x - \frac{ay}{\sqrt{1 - a^2}} + d_a\phi(a).$$

If a be eliminated between the two latter equations, the general primitive results.

This is a simple case of the problem of finding surfaces of equivalent area to a given surface, that is, so that any cylindrical surface parallel to axis of z may always intercept equal areas in the required and given surfaces.

If P, Q, be the values of d_xz, d_yz in the given surface, the general condition is

$$p^2 + q^2 = P^2 + Q^2$$

which, if the given surface be a plane, leads to the equation of this example.

Ex. 2. $$z = pq,$$

$$\frac{z}{y + a} = x + \phi(a),$$

$$\frac{z}{(y + a)^2} = - d_a\phi(a).$$

Equations of the Second and Higher Orders.

97. Here, besides the coefficients p and q of the first order which may be involved together with x, y, and z, the equation must contain one or more of the coefficients of the second order r, s, t; so that in its most general form it will be

$$F(x, y, z, p, q, r, s, t) = 0.$$

In partial differential equations of the second order, we cannot be certain of the form of the solution, nor pronounce

beforehand how many arbitrary functions it ought to contain. For let

$$u = f\{x,\ y,\ z,\ \phi(v),\ \psi(w)\} = 0,$$

be an equation containing two arbitrary functions of v and w two known functions of x, y, and z; then p and q will be given by the equations

$$d_x u = 0, \quad d_y u = 0;$$

and r, s, and t by the equations

$$d_x^2 u = 0, \quad d_x d_y u = 0, \quad d_y^2 u = 0.$$

These together with $u = 0$ make six equations, into which the six quantities

$$\phi(v), \quad \psi(w), \quad \phi'(v), \quad \psi'(w), \quad \phi''(v), \quad \psi''(w),$$

may enter. Consequently, it will not generally be possible to eliminate these six quantities and obtain a relation between x, y, z, p, q, r, s, t, independent of the forms of the functions ϕ and ψ; although particular cases do occur in which this can be effected.

In general, if $u = 0$ contain n independent variables, and m functions of the form $\phi(v)$, where v is a determinate function of the variables; these functions can certainly be eliminated between the equations obtained after (r) differentiations, if

$$1 + n + \frac{n(n+1)}{1.2} + \ldots + \frac{n(n+1)\ldots(n+r-1)}{\underline{|r}} > m(r+1),$$

the former expressing the number of equations, and the latter the number of functions.

98. As an example of forming a partial differential equation of the second order, by the elimination of two arbitrary functions, we may take

$$u = y - x\phi(z) - \psi(z) = 0;$$

$$\therefore\ d_x u = -\phi(z) - x\phi'(z)\,p - \psi'(z)\,p = 0,$$

$$d_y u = \quad 1 - x\phi'(z)\,q - \psi'(z)\,q = 0;$$

$$\therefore\ p + q\phi(z) = 0;$$

$$\therefore\ r + s\phi(z) + pq\phi'(z) = 0,$$

$$s + t\phi(z) + q^2\phi'(z) = 0;$$

$$\therefore\ qr - ps + (qs - pt)\phi(z) = 0,$$

$$\text{or}\quad q^2r - 2pqs + p^2t = 0.$$

99. The equations

$$d_x^2 z + Pd_x z + Q = 0,$$

$$d_y^2 z + Pd_y z + Q = 0,$$

where P and Q are functions of x, y, and z, must be integrated as equations between two variables, y being regarded as constant in the former, and x in the latter; and arbitrary functions of those variables respectively, being introduced instead of constants.

100. The equations

$$d_x d_y z + Pd_x z = Q,$$

$$d_x d_y z + Pd_y z = Q,$$

where P and Q do not contain z, are reducible to the case of Art. 90, by considering $d_x z$ or $d_y z$ respectively as a single quantity v.

Ex. $$d_x d_y z + \frac{y}{1-y^2} d_x z = ay^3.$$

This being a linear equation in $d_x z$ which is made integrable by the factor $\dfrac{1}{\sqrt{1-y^2}}$, we have

$$\frac{d_x z}{\sqrt{1-y^2}} = a\int_y \frac{y^3}{\sqrt{1-y^2}} = -a\left\{y^2\sqrt{1-y^2} + \tfrac{2}{3}(1-y^2)^{\frac{3}{2}}\right\} + \psi'(x);$$

$$\therefore\ z = \{\psi(x) + \phi(y)\}\sqrt{1-y^2} - \frac{ax}{3}(2+y^2)(1-y^2).$$

101. Similarly, equations of the forms

$$f(x,\ y,\ z,\ d_x z,\ d_x^2 z,\ \text{\&c.},\ d_x^n z) = 0,$$

$$f(x,\ y,\ z,\ d_y z,\ d_y^2 z,\ \text{\&c.},\ d_y^n z) = 0,$$

may be treated as if they contained only two variables; arbitrary functions of y instead of constants being introduced into the solution of the first, and arbitrary functions of x into the solution of the second. To this case may also be reduced the equation

$$f(x,\ y,\ d_y^n z,\ d_x d_y^n z,\ d_x^2 d_y^n z,\ \text{\&c.},\ d_x^m d_y^n z) = 0;$$

for by putting $d_y^n z = v$, it becomes

$$f(x,\ y,\ v,\ d_x v,\ d_x^2 v,\ \text{\&c.},\ d_x^m v) = 0,$$

which will give a value of v containing m arbitrary functions of y; and then $d_y^n z = v$ will give z involving n arbitrary functions of x.

102. To integrate the linear equation of the second order,

$$Rr + Ss + Tt = V,$$

where R, S, T, V are functions of x, y, z, p, q.

By means of the relations

$$dp = rdx + sdy,$$

$$dq = sdx + tdy;$$

eliminating two of the three coefficients, r and t, from the proposed, we get

$$Rdpdy + Tdqdx - Vdxdy = s\{R(dy)^2 + T(dx)^2 - Sdxdy\},$$

which is satisfied by

$$\left.\begin{aligned} Rdpdy + Tdqdx - Vdxdy &= 0 \\ R(dy)^2 + T(dx)^2 - Sdxdy &= 0 \end{aligned}\right\} \quad (1).$$

Let $M = a$, $N = b$, be two relations between x, y, z, p, q, and the arbitrary constants a, b, which satisfy these equations; then $M = \phi(N)$ satisfies the proposed equation. This will be shewn by proving that it can reproduce the proposed equation.

$$\text{Let } dy = mdx; \qquad \therefore\ Rm^2 - Sm + T = 0.$$

For each root of this equation, we have

$$dy - mdx = 0, \qquad Rmdp + Tdq - Vmdx = 0;$$

$$\left.\begin{aligned} \therefore\ dy &= mdx, \\ dq &= \frac{Vm}{T}\,dx - \frac{Rm}{T}\,dp, \\ dz &= pdx + qdy. \end{aligned}\right\} \quad (2).$$

Hence $M = a$ gives on differentiation

$$0 = d_xM\,.\,dx + d_yM\,.\,mdx + d_zM\,(pdx + qmdx)$$

$$+ d_pM\,.\,dp + d_qM\,.\left(\frac{Vm}{T}\,dx - \frac{Rm}{T}\,dp\right),$$

wherein all the known relations (2) having been introduced, dx and dp must be independent,

$$\therefore\ 0 = d_xM + md_yM + d_zM\,.\,(p + mq) + \frac{Vm}{T}\,d_qM,$$

$$0 = d_pM - \frac{Rm}{T}\,d_qM;$$

$$\therefore \quad d_x M = -\left\{m d_y M + (p + mq)\, d_z M + \frac{Vm}{T}\, d_q M\right\},$$

$$d_p M = \frac{Rm}{T}\, d_q M.$$

So $$d_x N = -\left\{m d_y N + (p + mq)\, d_z N + \frac{Vm}{T}\, d_q N\right\},$$

$$d_p N = \frac{Rm}{T}\, d_q N.$$

By differentiating the assumed equation $M = \phi(N)$ we have

$$dM = \phi'(N)\,.\,dN.$$

Now

$$dM = -\left\{m d_y M + (p + mq) d_z M + \frac{Vm}{T}\, d_q M\right\} dx + d_y M dy$$

$$+ d_z M\,(p dx + q dy) + \frac{Rm}{T}\, d_q M . dp + d_q M \,.\, dq$$

$$= (d_y M + q d_z M)\,(dy - m dx) + \frac{d_q M}{T}\,(Rm dp + T dq - Vm dx),$$

and a similar value exists for dN;

$$\therefore \quad (d_y M + q d_z M)\,(dy - m dx) + \frac{d_q M}{T}\,(Rm dp + T dq - Vm dx)$$

$$= \phi'(N)\left\{(d_y N + q d_z N)\,(dy - m dx)\right.$$

$$\left. + \frac{d_q N}{T}\,(Rm dp + T dq - Vm dx)\right\},$$

which may be put under the form

$$Rm dp + T dq - Vm dx = \omega\,(dy - m dx)$$

or $Rm(rdx + sdy) + T(sdx + tdy) - Vmdx = \omega(dy - mdx)$,

where dx and dy are independent;

$$\therefore\ Rmr + Ts - Vm = -\omega m,$$

$$Rms + Tt = \omega;$$

$$\therefore\ Rr + Ss + Tt = V - \frac{s}{m}(Rm^2 - Sm + T) = V.$$

Hence, $M = \phi(N)$ satisfies the proposed equation.

According as the roots of

$$Rm^2 - Sm + T = 0$$

are unequal or equal, we are thus supplied with a total or partial differential equation for the determination of z.

Obs. As the reducing equations (1) may contain

$$x,\ y,\ z,\ p,\ q,$$

and as these together with $dz = pdx + qdy$ will generally lead to an equation containing three variables, which will not always admit of a single primitive (Art. 83.), it may happen that the first integral of the proposed equation cannot be determined; but we must not thence conclude that the proposed equation does not admit of being solved.

103. Hence, to integrate the linear equation of the second order

$$Rr + Ss + Tt = V,$$

the process is to obtain a value of m from the equation

$$Rm^2 - Sm + T = 0,$$

to substitute it in the system

$$dy - mdx = 0, \quad Rmdp + Tdq = Vmdx, \qquad (1.)$$

to satisfy these, conjointly or separately, by two relations between x, y, z, p, q,

$$M = a, \quad N = b,$$

then to put $M = \phi(N)$, and to integrate this equation of the first order.

104. If R, S, T, be constant, and V a function of x and y only, then the values of m will be numerical, m and n suppose; and the integrals of equations (1) will be

$$y - mx = a, \quad Rmp + Tq = m\int_x V + b,$$

where, previous to integration, $mx + a$ is substituted for y in V, and after integration the value of a, viz. $y - mx$, is restored: consequently calling this value V_1, since $Rmn = T$, we have

$$Rp + Rnq = V_1 + \phi'(y - mx).$$

Next to integrate this equation of the first order, we have the reducing equations

$$dy - n\,dx = 0,$$

$$R\,dz - \{V_1 + \phi'(y - mx)\}\,dx = 0;$$

$$\therefore\ y - nx = a,$$

$$Rz = \int_x V_1 + \int_x \phi'(y - mx) + \beta,$$

$nx + a$ being substituted for y before the integration is performed, and afterwards the value of a, $y - nx$, restored; this gives $\int_x V_1 = V_2$, suppose,

and $\int_x \phi'(y - mx) = \int_x \phi'\{(n - m)x + a\} = \dfrac{1}{n - m}\phi(y - mx)$;

hence, including the constant multiplier under the sign of the function,

$$Rz = V_2 + \phi(y - mx) + \psi(y - nx).$$

Ex. 1. $\quad d_x^2 z = a^2 d_y^2 z, \quad$ or $\ r - a^2 t = 0,$

$$\text{or } \ r\,dx\,dy - a^2 t\,dy\,dx = 0,$$

$$\text{or} \quad dpdy - s(dy)^2 - a^2\{dqdx - s(dx)^2\} = 0;$$

$$\therefore \quad (dy)^2 = a^2(dx)^2,$$

$$dpdy = a^2dqdx,$$

$$dy = \pm\, adx.$$

1st. $\quad dy = adx, \qquad y - ax = \alpha,$

$$dp = adq, \qquad p - aq = \beta = f'(y - ax). \qquad (1).$$

2nd. $\quad dy = -\, adx, \qquad y + ax = \alpha,$

$$dp = -\, adq, \qquad p + aq = \beta = \phi'(y + ax). \qquad (2).$$

By adding and subtracting (1) and (2),

$$2p = f'(y - ax) + \phi'(y + ax),$$

$$2aq = \phi'(y + ax) - f'(y - ax);$$

$$\therefore \quad 2adz = 2apdx + 2aqdy = \phi'(y + ax)(dy + adx)$$
$$- f'(y - ax)(dy - adx);$$

$$\therefore \quad z = \phi(y + ax) + f(y - ax).$$

Ex. 2. $$r - \frac{2p}{q}s + \frac{p^2}{q^2}t = 0,$$

$$rdxdy - \frac{2p}{q}sdxdy + \frac{p^2}{q^2}tdydx = 0,$$

$$dpdy - s(dy)^2 - \frac{2p}{q}sdxdy + \frac{p^2}{q^2}\{dqdx - s(dx)^2\} = 0;$$

$$\therefore \quad (dy)^2 + 2\frac{p}{q}dydx + \frac{p^2}{q^2}(dx)^2 = 0,$$

$$dpdy + \frac{p^2}{q^2}dqdx = 0;$$

$$\therefore \; dy = -\frac{p}{q}\,dx,$$

$$dp - \frac{p}{q}\,dq = 0,$$

$$dz = pdx + qdy = 0\,, \qquad \therefore \; z = a,$$

$$\frac{qdp - pdq}{q^2} = 0, \qquad \frac{p}{q} = b\,;$$

$$\therefore \; p - qf(z) = 0.$$

To integrate this equation of the first order, we have

$$dz = 0, \qquad \text{or} \quad z = a,$$

$$dy + f(z)\,dx = 0, \qquad \text{or} \quad y + xf(a) = \beta\,;$$

$$\therefore \; y + xf(z) = \phi\,(z)\,;$$

this is the equation to the surface generated by a straight line, subjected to pass through three given fixed curves.

Ex. 3. $$r - a^2 t = xy,$$

$$z = \frac{1}{6}\,x^3 y + f\,(y + ax) + \phi\,(y - ax).$$

4. $x^2 r - y^2 t = 0. \quad z = \sqrt{xy}\,.\,\phi\left(\dfrac{y}{x}\right) + \psi\,(xy).$

5. $r+3s+2t=x+y. \quad z=\frac{1}{2}\,x^2 y-\frac{1}{3}\,x^3+\phi\,(y-x)+\psi(y-2x).$

6. $x^2 r + 2xys + y^2 t = 0, \qquad z = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right).$

7. $r + (a + b)\,s + abt = xy,$

$$z - \frac{1}{6}\,x^3 y + \frac{1}{24}\,(a + b)x^4 = \phi\,(y - ax) + \psi\,(y - bx).$$

8. $(1 + pq + q^2)\,r + s\,(q^2 - p^2) - (1 + pq + p^2)\,t = 0.$

A TREATISE

ON THE

CALCULUS OF FINITE DIFFERENCES.

SECTION I.

DIRECT METHOD OF DIFFERENCES.

Definitions and Principles.

ART. 1. IN the Differential Calculus it is the first term only of the series, arranged according to ascending powers of h, expressing $u_{x+h} - u_x$ (where u_x is any function of x and u_{x+h} the same function of $x + h$), or rather the coefficient of h in that term, with which we are principally concerned, and which we usually write $hd_x u_x$. But in the Calculus of Finite Differences, it is the whole of that series which forms the object of our investigations, and it is usually written Δu_x, so that

$$\Delta u_x = u_{x+h} - u_x.$$

2. It is common, however, to suppose the finite increment which the principal variable x receives, to be 1 instead of h, both for the sake of simplicity, and because that is the value of the increment when u_x is regarded as the general term of a series; and in that light it is by far the most frequently regarded in finite differences; so that

$$\Delta u_x = u_{x+1} - u_x.$$

There are, however, in this subject, several important theorems which it is advantageous to investigate on the hypothesis of an indeterminate increment h, for the principal variable, instead of unity, as the process is the same on either supposition, and the result one of greater generality. And if in other cases it should be desirable to introduce the same hypothesis, the expression must be prepared by first writing hx instead of x, then performing the operation on the usual supposition of $\Delta x = 1$, and in the result writing $\frac{x}{h}$ for x.

3. By a Series is meant a regular progression of terms increasing or decreasing in magnitude according to a certain law; hence, when that law is given, and also the place of any term in the series, the magnitude of the term may be found, and thus the successive terms of the series may be produced in order. The place of any term in a series is assigned by giving the number of terms by which it is removed from some one which is considered as fixed. This number is called the index of the term to which it belongs. Thus in the series

$$0, \ 1, \ 8, \ 27, \ 64, \ \ldots \ x^3, \ldots$$

taking the first term as the point of departure, we have the corresponding series of indices

$$0, \ 1, \ 2, \ 3, \ 4, \ \ldots \ x, \ \ldots$$

If the series be continued backwards, the indices must be considered as negative; thus the backward continuation of the above series gives the terms

$$\ldots -x^3, \ \ldots -27, \ -8, \ -1,$$

with the corresponding indices

$$-x, \ \ldots -3, \ -2, \ -1.$$

4. Since the magnitude of every term is determined solely by its index and by the law of the series, it follows that any term is a certain function of its index, the form of which does not alter in passing from one term to another, but remains the same throughout the whole series. Thus in the above series every term is the cube of its index.

This function analytically expressed is called the general term of the series; (in the above series the general term is x^3;) and it is evident that all the terms of the series will be produced from it in order, by substituting successively for the index x, the progression of natural numbers

$$\ldots -2,\ -1,\ 0,\ 1,\ 2, \ldots$$

The general term of a series is usually denoted by u_x, where u_x is a certain function of x determined by the nature of the series. Thus, u_x denoting the general term, the series will be

$$\ldots u_{-2},\ u_{-1},\ u_0,\ u_1,\ u_2,\ \ldots u_{x-1},\ u_x,\ u_{x+1},\ \ldots$$

5. The excess of any term u_{x+1} above that which immediately precedes it, or the function $u_{x+1} - u_x$ is called, as has been stated, the Difference of the function u_x, and is denoted by Δu_x. (In certain cases, which will however be expressly mentioned, we shall take Δu_x to mean $u_{x+h} - u_x$.)

It is obvious that $u_{x+1} - u_x$ is itself in general a certain function of x, the nature of which is entirely dependent on that of the original function u_x, from which it is derived, and susceptible of a difference.

The difference, consequently, of the function Δu_x (which must be considered as having Δu for its characteristic, in the same manner as u_x has u) is

$$\Delta(\Delta u_x) = \Delta u_{x+1} - \Delta u_x,$$

which is usually written $\Delta^2 u_x$.

In like manner

$$\Delta(\Delta^2 u_x) = \Delta^3 u_x = \Delta^2 u_{x+1} - \Delta^2 u_x,$$

$$\Delta(\Delta^3 u_x) = \Delta^4 u_x = \Delta^3 u_{x+1} - \Delta^3 u_x,$$

....

$$\Delta^n u_x = \Delta^{n-1} u_{x+1} - \Delta^{n-1} u_x.$$

6. Hence if in any function of x we change x into $x + 1$, and from the result subtract the proposed function, we obtain the first difference of the proposed function; and the second, third, &c. differences are formed, each from the preceding, by a similar operation. To determine these differences of given functions, and to investigate the relations which hold between differences of any orders and the functions from which they are derived, is the object of the direct method of differences.

We shall now proceed to give instances of finding the differences of various functions, according to the above definition.

Differences of Explicit Functions.

7. To find the difference of $au_x + c$, u_x being any function of x, and a and c quantities independent of x.

$$\Delta(au_x + c) = au_{x+1} + c - (au_x + c) = a(u_{x+1} - u_x) = a\Delta u_x.$$

Hence, making $a = 0$, $\Delta c = 0$.

8. To find the difference of the sum of any number of functions of x.

$$\Delta(u_x + v_x + w_x) = u_{x+1} + v_{x+1} + w_{x+1} - (u_x + v_x + w_x)$$

$$= u_{x+1} - u_x + v_{x+1} - v_x + w_{x+1} - w_x$$

$$= \Delta u_x + \Delta v_x + \Delta w_x.$$

9. To find the difference of the product of two functions.

$$\Delta (u_x v_x) = u_{x+1} v_{x+1} - u_x v_x = (u_x + \Delta u_x)(v_x + \Delta v_x) - u_x v_x$$

$$= u_x \Delta v_x + v_x \Delta u_x + \Delta u_x . \Delta v_x = u_x \Delta v_x + v_{x+1} \Delta u_x .$$

10. To find the difference of the quotient of two functions.

$$\Delta \left(\frac{u_x}{v_x}\right) = \frac{u_{x+1}}{v_{x+1}} - \frac{u_x}{v_x} = \frac{(u_x + \Delta u_x) v_x - (v_x + \Delta v_x) u_x}{v_{x+1} v_x}$$

$$= \frac{v_x \Delta u_x - u_x \Delta v_x}{v_{x+1} v_x} .$$

11. To find the difference of the continued product of any number of successive values of a function.

$$\Delta (u_x u_{x+1} \dots u_{x+n}) = u_{x+1} u_{x+2} \dots u_{x+n+1} - u_x u_{x+1} \dots u_{x+n}$$

$$= u_{x+1} u_{x+2} \dots u_{x+n} (u_{x+n+1} - u_x).$$

Hence in the particular case where $u_x = a + bx$

$$\Delta (u_x u_{x+1} \dots u_{x+n}) = u_{x+1} u_{x+2} \dots u_{x+n} . (n + 1) b.$$

12. To find the difference of a fraction whose numerator and denominator are the continued products of any number of successive values of two functions u_x and v_x respectively.

$$\Delta \left(\frac{u_x u_{x+1} \dots u_{x+n}}{v_x v_{x+1} \dots v_{x+m}}\right) = \frac{u_{x+1} u_{x+2} \dots u_{x+n}}{v_x v_{x+1} \dots v_{x+m+1}} (v_x u_{x+n+1} - u_x v_{x+m+1}).$$

Hence $$\Delta \frac{1}{v_x v_{x+1} \dots v_{x+m}} = - \frac{v_{x+m+1} - v_x}{v_x v_{x+1} \dots v_{x+m+1}} ,$$

and in the particular case where $v_x = a + bx$,

$$\Delta \frac{1}{v_x v_{x+1} \dots v_{x+m}} = - \frac{(m + 1) b}{v_x v_{x+1} \dots v_{x+m+1}} .$$

13. To find the differences of any rational integral function, and to shew that the n^{th} difference of a rational integral function of the n^{th} degree, is constant.

Let $u_x = Ax^n + Bx^{n-1} + \ldots + Kx + L$, be a rational integral function; then its first difference is

$$\Delta u_x = A\{(x+1)^n - x^n\} + B\{(x+1)^{n-1} - x^{n-1}\} + \ldots + I(2x+1) + K$$

$$= nAx^{n-1} + B_1 x^{n-2} + \ldots + I_1 x + K_1,$$

which is a rational integral function, one degree lower than the original function. In like manner, for the difference of this, or the second difference of u_x, we have

$$\Delta^2 u_x = n(n-1)Ax^{n-2} + B_2 x^{n-3} + \ldots + I_2,$$

and so on; and for the n^{th} difference, we have

$$\Delta^n u_x = n(n-1)\ldots 3.2.1.A.$$

Hence the n^{th} difference is constant, and the differences of all orders superior to the n^{th} vanish.

Also $\Delta^n (x^n) = 1.2.3\ldots n.$

14. To find the differences of a^x.

$$\Delta a^x = a^{x+1} - a^x = a^x(a-1),$$

$$\Delta^2 a^x = (a-1)\Delta a^x = a^x(a-1)^2,$$

$$\Delta^n a^x = a^x(a-1)^n.$$

Also $\Delta a^{v_x} = a^{v_x + \Delta v_x} - a^{v_x} = a^{v_x}(a^{\Delta v_x} - 1).$

15. To find the difference of $\log v_x$.

$$\Delta(\log v_x) = \log v_{x+1} - \log v_x = \log \frac{v_{x+1}}{v_x} = \log\left(1 + \frac{\Delta v_x}{v_x}\right).$$

16. To find the differences of $\sin v_x$ and $\cos v_x$.

$$\Delta \sin v_x = \sin(v_x + \Delta v_x) - \sin v_x = 2 \sin \frac{\Delta v_x}{2} \cos\left(v_x + \frac{\Delta v_x}{2}\right)$$

$$= 2 \sin \frac{\Delta v_x}{2} \sin\left\{v_x + \tfrac{1}{2}(\pi + \Delta v_x)\right\}.$$

$$\Delta \cos v_x = \cos(v_x + \Delta v_x) - \cos v_x = -2 \sin \frac{\Delta v_x}{2} \sin\left(v_x + \frac{\Delta v_x}{2}\right)$$

$$= 2 \sin \frac{\Delta v_x}{2} \cos\left\{v_x + \tfrac{1}{2}(\pi + \Delta v_x)\right\}.$$

Hence $$\Delta^n \sin(x\theta + a) = \left(2 \sin \frac{\theta}{2}\right)^n \sin\left\{x\theta + a + \frac{n}{2}(\pi + \theta)\right\},$$

$$\Delta^n \cos(x\theta + a) = \left(2 \sin \frac{\theta}{2}\right)^n \cos\left\{x\theta + a + \frac{n}{2}(\pi + \theta)\right\}.$$

17. To find the differences of $\tan v_x$ and $\tan^{-1} v_x$.

$$\Delta \tan v_x = \tan v_{x+1} - \tan v_x = \frac{\sin v_{x+1} \cos v_x - \cos v_{x+1} \sin v_x}{\cos v_{x+1} \cos v_x}$$

$$= \frac{\sin(v_{x+1} - v_x)}{\cos v_{x+1} \cos v_x} = \frac{\sin \Delta v_x}{\cos v_{x+1} \cos v_x}.$$

Hence $$\Delta \tan x\theta = \frac{\sin \theta}{\cos(x+1)\theta \cos x\theta}.$$

Also $$\Delta \tan^{-1} v_x = \tan^{-1} v_{x+1} - \tan^{-1} v_x = \tan^{-1} \frac{v_{x+1} - v_x}{1 + v_{x+1} v_x}$$

$$= \tan^{-1} \frac{\Delta v_x}{1 + v_{x+1} v_x}.$$

Hence $$\Delta \tan^{-1} x\theta = \tan^{-1} \frac{\theta}{1 + (x+1)x\theta^2}.$$

Relations between the successive Values and the Differences of a Function.

18. To express $\Delta^n u_x$ by u_x and its n successive values, u_{x+h}, u_{x+2h}, ... u_{x+nh}.

We here take h instead of unity for the increment of the principal variable, as the investigation is precisely the same on either supposition.

$$\Delta u_x = u_{x+h} - u_x$$

$$\Delta^2 u_x = u_{x+2h} - u_{x+h} - (u_{x+h} - u_x) = u_{x+2h} - 2u_{x+h} + u_x,$$

$$\Delta^3 u_x = u_{x+3h} - 2u_{x+2h} + u_{x+h} - (u_{x+2h} - 2u_{x+h} + u_x),$$

$$= u_{x+3h} - 3u_{x+2h} + 3u_{x+h} - u_x.$$

Now suppose this law of the coefficients, which as far as we have gone is the same as that of an expanded binomial whose index is the order of the difference, to hold for the n^{th} difference, so that

$$\Delta^n u_x =$$

$$u_{x+nh} - p_1 u_{x+(n-1)h} + p_2 u_{x+(n-2)h} - \dots \mp p_1 u_{x+h} \pm u_x,$$

then $\Delta^{n+1} u_x =$

$$u_{x+(n+1)h} - p_1 u_{x+nh} + p_2 u_{x+(n-1)h} - \dots \mp p_1 u_{x+2h} \pm u_{x+h}$$

$$- (u_{x+nh} - p_1 u_{x+(n-1)h} + \dots \pm p_2 u_{x+2h} \mp p_1 u_{x+h} \pm u_x)$$

$$= u_{x+(n+1)h} - (1 + p_1)\, u_{x+nh} + (p_1 + p_2)\, u_{x+(n-1)h} - \dots$$

$$\pm (1 + p_1)\, u_{x+h} \mp u_x,$$

which is the same alteration with regard to the coefficients as occurs in passing from $(x-1)^n$ to $(x-1)^{n+1}$. If therefore, for any value of n supposed a positive integer, the coefficients of the expansions of $\Delta^n u_x$ and $(x-1)^n$ are the same, they will always be the same; but these coefficients are identical when $n = 1, 2, 3$; therefore they are always the same;

$$\therefore\ \Delta^n u_x = u_{x+nh} - n u_{x+(n-1)h} + \frac{n(n-1)}{1.2} u_{x+(n-2)h} - \dots$$

$$\mp n u_{x+h} \pm u_x;$$

or, supposing $h = 1$,

$$\Delta^n u_x = u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \dots \mp n u_{x+1} \pm u_x.$$

Obs. For the sake of being easily remembered, this result may be written

$$\Delta^n u_x = (u_x - 1)^n;$$

where, instead of the powers of u_x from n to 0, the $n+1$ descending values $u_{x+n}, u_{x+n-1}, \dots u_{x+1}, u_x$ are to be written.

19. If in the series just investigated, we assign a particular value to u_x, we shall readily obtain an expression for its n^{th} difference. Thus let $u_x = x^m$,

$$\therefore\ \Delta^n(x^m) = (x+n)^m - n(x+n-1)^m + \frac{n(n-1)}{1.2}(x+n-2)^m - \&c.,$$

in which equation, if $n > m$, since the former member vanishes, the second member is zero for every value of x; and if $m = n$, so that $\Delta^n x^n = 1.2.3 \dots n$, we get

$$1.2.3\dots n = (x+n)^n - n(x+n-1)^n + \frac{n(n-1)}{1.2}(x+n-2)^n - \&c.;$$

and making $x = 0$, since the equation holds for all values of x,

$$1.2.3\dots n = n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \&c.$$

20. If $x=0$, and $\Delta^n 0^m$ denote the particular value of $\Delta^n x^m$ when $x=0$, we have

$$\Delta^n 0^m = n^m - n(n-1)^m + \frac{n(n-1)}{1.2}(n-2)^m - \&c.$$

Of the numbers comprised in the form $\Delta^n 0^m$, we shall make considerable use in future investigations; whenever $n > m$ the value is zero, in other cases it may be computed by the above formula; thus,

$$\Delta 0^m = 1,$$

$$\Delta^2 0^2 = 2, \qquad \Delta^2 0^3 = 6, \qquad \Delta^2 0^4 = 14, \ldots$$

$$\Delta^3 0^3 = 6, \qquad \Delta^3 0^4 = 36, \qquad \Delta^3 0^5 = 150, \ldots$$

21. Reversing the order of the series in Art. 18, we find

$$(-1)^n \Delta^n u_x = u_x - nu_{x+1} + \frac{n(n-1)}{1.2} u_{x+2} - \ldots \pm u_{x+n}.$$

Also putting $u_x = x^m$, and then supposing $m=n$, $x=1$, we find successively,

$$(-1)^n \Delta^n x^m = x^m - n(x+1)^m + \frac{n(n-1)}{1.2}(x+2)^m - \ldots \pm (x+n)^m,$$

$$(-1)^n 1.2.3..n = x^n - n(x+1)^n + \frac{n(n-1)}{1.2}(x+2)^n - \ldots \pm (x+n)^n,$$

$$(-1)^n 1.2.3..n = 1^n - n.2^n + \frac{n(n-1)}{1.2}.3^n - \ldots \pm (n+1)^n.$$

22. Hence it may be proved that $1.2.3\ldots(p-1)+1$ is divisible by p, if p be a prime number.

Let $u_x = x^n - 1$;

$$\therefore\ \Delta^n(x^n-1)=1.2.3\ldots n=(x+n)^n-1-n\left\{(x+n-1)^n-1\right\}$$
$$+\frac{n(n-1)}{1.2}\left\{(x+n-2)^n-1\right\}-\&c.$$

Let $x=1$, and $n+1=p$, then $x+n=p$, and

$$1.2.3\ldots(p-1)+1=p^n-(p-1)\left\{(p-1)^{p-1}-1\right\}$$
$$+\frac{(p-1)(p-2)}{1.2}\left\{(p-2)^{p-1}-1\right\}-\&c.$$

Now by Fermat's Theorem every term of the second member is divisible by p when p is a prime number; consequently $1.2.3\ldots(p-1)+1$ is divisible by p.

23. To express u_{x+nh} by u_x and its first n differences.

We take h instead of unity for the increment of the principle variable, the investigation being precisely the same on either supposition.

$$u_{x+h}=u_x+\Delta u_x,$$
$$u_{x+2h}=u_x+\Delta u_x+\Delta(u_x+\Delta u_x)=u_x+2\Delta u_x+\Delta^2 u_x,$$
$$u_{x+3h}=u_x+2\Delta u_x+\Delta^2 u_x+\Delta(u_x+2\Delta u_x+\Delta^2 u_x)$$
$$=u_x+3\Delta u_x+3\Delta^2 u_x+\Delta^3 u_x.$$

Now suppose this law of coefficients, which as far as we have gone is the same as that of an expanded binomial, whose index is the number of increments which the principal variable has received, to hold for n increments, so that

$$u_{x+nh}=u_x+p_1\Delta u_x+p_2\Delta^2 u_x+\ldots+p_1\Delta^{n-1}u_x+\Delta^n u_x,$$

then $u_{x+(n+1)h}=u_x+p_1\Delta u_x+p_2\Delta^2 u_x+\ldots+p_1\Delta^{n-1}u_x+\Delta^n u_x$

$$+\Delta(u_x+p_1\Delta u_x+\ldots+p_2\Delta^{n-2}u+p_1\Delta^{n-1}u_x+\Delta^n u_x)$$
$$=u_x+(1+p_1)\Delta u_x+(p_1+p_2)\Delta^2 u_x+\ldots$$
$$+(p_2+p_1)\Delta^{n-1}u_x+(p_1+1)\Delta^n u_x+\Delta^{n+1}u_x,$$

which is the same alteration with regard to the coefficients as occurs in passing from $(1+z)^n$ to $(1+z)^{n+1}$.

Hence if the coefficients of the expansions of u_{x+nh} and $(1+z)^n$ are the same for any value of n, supposed a positive integer, they will always be the same; but they are identical when $n = 1, 2, 3$; therefore they are always the same;

$$\therefore \quad u_{x+nh} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \ldots + n\Delta^{n-1}u_x + \Delta^n u_x,$$

or, supposing $h = 1$,

$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \ldots + n\Delta^{n-1}u_x + \Delta^n u_x.$$

Obs. This result in order to be more easily remembered may be written

$$u_{x+n} = (1+\Delta)^n u_x,$$

each term of the development of $(1+\Delta)^n$ being understood to be prefixed to u_x.

24. Let $u_x = x^m$, then $u_{x+n} = (x+n)^m$;

$$\therefore \quad (x+n)^m = (1+\Delta)^n x^m,$$

$$\text{and making } x = 0, \quad n^m = (1+\Delta)^n 0^m,$$

each term of the development of $(1+\Delta)^n$ being prefixed, as said above, to x^m and 0^m, respectively. By the latter formula any power of a number is expressed by the numbers comprised in the form $\Delta^n 0^m$.

25. To deduce Taylor's Theorem from the formula

$$u_{x+nh} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \&c.$$

Let $nh = t$, and let h be infinitely diminished whilst t remains finite; therefore n is infinitely increased; and since

h is indefinitely diminished, we have, regarding the differential coefficient as the limit of the ratio of the simultaneous increments of the function and the variable,

$$\frac{\Delta u_x}{h} = d_x u_x,$$

$$\frac{\Delta^2 u_x}{h^2} = d_x \left(\frac{\Delta u_x}{h}\right) = d_x^2 u_x; \text{ \&c.}$$

Hence, preparing the formula as follows,

$$u_{x+nh} = u_x + nh \frac{\Delta u_x}{h} + \frac{nh(nh-h)}{1.2} \frac{\Delta^2 u_x}{h^2} + \text{\&c.}$$

and taking the limit of both sides by supposing h to be infinitely diminished and n infinitely increased, their product always remaining equal to a finite magnitude t, we get

$$u_{x+t} = u_x + t d_x u_x + \frac{t^2}{1.2} d_x^2 u_x + \text{\&c.}$$

The Differential Calculus is a particular case of that of Finite Differences; and the above investigation is introduced to shew how, from results in Finite Differences obtained with an indeterminate increment for the principal variable, we may pass to the corresponding results in the Differential Calculus.

26. The theorems of Arts. 18 and 23 have been proved by an inductive process; they may also be established by the theory of Generating Functions, the principles of which we shall now proceed to explain; as it is a theory which, for its generality and power, especially merits our attention.

Generating Functions.

27. Let $\phi(t)$ be a function of t susceptible of the development

$$\phi(t) = \ldots + u_{-1}t^{-1} + u_0 + u_1 t + \ldots$$
$$+ u_{x-1}t^{x-1} + u_x t^x + u_{x+1}t^{x+1} + \ldots;$$

then u_x may evidently represent any function of x whatever, if we regard this equation as the definition of $\phi(t)$. The

function $\phi(t)$ consequently by its development generates the coefficients u_0, u_1, ... u_x annexed to their proper powers of t, and is therefore called the Generating Function of u_x, and is denoted by Gu_x, so that

$$\phi(t) = Gu_x.$$

Thus since

$$\log\,(1-t)^{-1} = t + \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 + \dots + \frac{1}{x}t^x + \dots$$

$$\log\,(1-t)^{-1} = G\frac{1}{x}.$$

Similarly, since

$$t(1-t)^{-2} = t + 2t^2 + 3t^3 + \dots + xt^x + \dots$$

$$t(1-t)^{-2} = Gx.$$

28. To determine the generating functions of u_{x+n} and $\Delta^n u_x$ from that of u_x.

Let $\phi(t) = Gu_x$,

then $\phi(t) = \quad \dots + u_n t^n + u_{n+1}t^{n+1} + \dots + u_{x+n}t^{x+n} + \dots$

$$\therefore\ t^{-n}\phi(t) = \dots + u_n + u_{n+1}t + \dots + u_{x+n}t^x + \dots$$

$$\therefore\ t^{-n}\phi(t) = Gu_{x+n}, \quad \text{or } t^{-n}Gu_x = Gu_{x+n}.$$

Again, $t^n\phi(t) = \dots + u_{-n} + u_{-n+1}t + \dots + u_{x-n}t^x + \dots$

$$\therefore\ t^n\phi(t) = Gu_{x-n}, \quad \text{or } t^nGu_x = Gu_{x-n}.$$

Hence it follows that the generating function of

$$\Delta u_x, \quad \text{or } u_{x+1} - u_x, \quad \text{is } \left(\frac{1}{t} - 1\right)\phi(t),$$

for this function being developed will produce the difference of two series whose general terms are respectively $u_{x+1}t^x$ and u_xt^x;

$$\therefore\ G(\Delta u_x) = \left(\frac{1}{t} - 1\right) Gu_x.$$

Similarly,

$$G(\Delta^2 u_x) = \left(\frac{1}{t} - 1\right) G(\Delta u_x) = \left(\frac{1}{t} - 1\right)^2 Gu_x,$$

$$\text{and}\ G(\Delta^n u_x) = \left(\frac{1}{t} - 1\right)^n Gu_x.$$

Also

$$G(\Delta^n u_{x-n}) = \left(\frac{1}{t} - 1\right)^n Gu_{x-n} = \left(\frac{1}{t} - 1\right)^n t^n Gu_x = (1-t)^n Gu_x.$$

29. **To investigate the expression for $\Delta^n u_x$ in terms of u_{x+n}, u_{x+n-1}, &c., by Generating Functions.**

$$G(\Delta^n u_x) = \left(\frac{1}{t} - 1\right)^n Gu_x$$

$$= t^{-n} Gu_x - nt^{-n+1} Gu_x + \frac{n(n-1)}{1.2} t^{-n+2} Gu_x - \&c.$$

$$= Gu_{x+n} - nGu_{x+n-1} + \frac{n(n-1)}{1.2} Gu_{x+n-2} - \&c.$$

$$= G\left(u_{x+n} - nu_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \&c.\right)$$

$$\therefore\ \Delta^n u_x = u_{x+n} - nu_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \&c.;$$

for, the generating functions of both being the same, the coefficients of t^x in their developments must be identical, however those developments have been effected.

30. To investigate the expression for u_{x+n} in terms of u_x and its first n differences, by Generating Functions.

$$Gu_{x+n} = t^{-n} Gu_x$$

$$= \left\{1 + \left(\frac{1}{t} - 1\right)\right\}^n Gu_x$$

$$= Gu_x + n\left(\frac{1}{t} - 1\right) Gu_x + \frac{n(n-1)}{1.2}\left(\frac{1}{t} - 1\right)^2 Gu_x + \&c.$$

$$= Gu_x + nG(\Delta u_x) + \frac{n(n-1)}{1.2} G(\Delta^2 u_x) + \&c.$$

$$= G\left(u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \&c.\right);$$

$$\therefore\ u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \&c.$$

31. It is obvious that by transforming the expressions

$$\left(\frac{1}{t} - 1\right)^n Gu_x, \quad \text{and } t^{-n} Gu_x,$$

in different ways, we may obtain various other expressions for $\Delta^n u_x$ and u_{x+n} besides the above.

Thus to express $\Delta^n u_x$ in terms of $\Delta^n u_{x-n}$, $\Delta^{n+1} u_{x-n-1}$, &c., we have

$$G(\Delta^n u_x) = \left(\frac{1}{t} - 1\right)^n Gu_x = \frac{(1-t)^n Gu_x}{\{1-(1-t)\}^n}$$

$$= (1-t)^n Gu_x + n(1-t)^{n+1} Gu_x$$

$$+ \frac{n(n+1)}{1.2}(1-t)^{n+2} Gu_x + \&c.$$

$$= G(\Delta^n u_{x-n}) + nG(\Delta^{n+1} u_{x-n-1}) + \frac{n(n+1)}{1.2} G(\Delta^{n+2} u_{x-n-2}) + \&c.$$

$$= G\left\{\Delta^n u_{x-n} + n\Delta^{n+1} u_{x-n-1} + \frac{n(n+1)}{1.2} \Delta^{n+2} u_{x-n-2} + \&c\right\};$$

$$\therefore \quad \Delta^n u_x =$$

$$\Delta^n u_{x-n} + n\Delta^{n+1} u_{x-n-1} + \frac{n(n+1)}{1.2} \Delta^{n+2} u_{x-n-2} + \&c.$$

32. Again, to express u_{x+n} in terms of u_x, Δu_{x-r}, $\Delta^2 u_{x-2r}$, &c., we must transform t^{-n} into a series of powers of $t^r\left(\frac{1}{t} - 1\right)$, or develope t^{-n} in powers of x from the equation

$$\frac{1}{t} = 1 + x\left(\frac{1}{t}\right)^r,$$

which may be done by Lagrange's Theorem; and we find

$$u_{x+n} = u_x + n\Delta u_{x-r} + \frac{n(n+2r-1)}{1.2} \Delta^2 u_{x-2r}$$

$$+ \frac{n(n+3r-1)(n+3r-2)}{1.2.3} \Delta^3 u_{x-3r} + \&c.$$

(Herschel's Examples).

Obs. This method is obviously not confined to the function $u_{x+1} - u_x$; it is equally applicable to any other combination of the successive values u_x, u_{x+1}, u_{x+2}, &c., of the first degree. If we take Δu_x to mean $au_{x+2} + bu_{x+1} + cu_x$, then the generating functions of Δu_x and $\Delta^n u_x$ will evidently be

$$\left(\frac{a}{t^2} + \frac{b}{t} + c\right) Gu_x, \qquad \left(\frac{a}{t^2} + \frac{b}{t} + c\right)^n Gu_x,$$

and the expression for $\Delta^n u_x$ in terms of u_x and its successive values, might be obtained as in the preceding case.

Separation of the Symbols of Operation from those of Quantity.

33. We have seen (Art. 23) in the formula

$$u_{x+n} = (1+\Delta)^n u_x$$

an instance of the system of notation, which consists in separating the symbols of operation from those of quantity; the use of which is not confined to simple cases like that just noticed, but may be extended with remarkable effect to a great variety of investigations connected with this subject.

If the expression $(1+\Delta)^n$, regarded as a function of a certain symbol Δ, be expanded in powers of Δ, it will produce the series,

$$1 + n\Delta + \frac{n(n-1)}{1.2}\Delta^2 + \&c.,$$

and if the symbol Δ be looked upon merely as an instrument by means of which we are enabled to produce the numerical coefficients of the series affected with their proper powers of Δ, then the expression $(1+\Delta)^n$ must be considered as having no other meaning than as an abbreviated expression for its development; and when prefixed to the function u_x, each term of this development is understood to be applied separately to that function; so that the following expressions,

$$(1+\Delta)^n u_x,$$

$$\left(1 + n\Delta + \frac{n(n-1)}{1.2}\Delta^2 + \ldots\right) u_x,$$

$$u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \ldots,$$

may be used indifferently for one another, the two former being regarded merely as abbreviations of the latter.

34. In general, if $F(\Delta)$ be a function of Δ capable of being developed in a series of powers of Δ, such as

$A\Delta^\alpha + B\Delta^\beta + \&c.$; then for $A\Delta^\alpha u_x + B\Delta^\beta u_x + \&c.$

the expression $F(\Delta)\,u_x$ is used as an abbreviation; and the same notation is applicable to other characteristic letters, such as d_x, $\int_x$, Σ, δ_x, &c.

Hence also, the successive performance of two or more series of operations represented by $F(\Delta)$, $F'(\Delta)$, &c. upon the same function u_x, is equivalent to the performance of that series of operations which is represented by their product.

For example, we have seen that

$$\Delta(u_x v_x) = \Delta u_x \,.\, v_x + u_x \Delta v_{x+1} = (\Delta + \Delta v)\, u_x v_x,$$

supposing Δ to affect u_x only, and using Δv for an operation affecting v_x only, such that $(\Delta v)^r v_x = \Delta^r v_{x+r}$;

then $\quad \Delta^2(u_x v_x) = (\Delta + \Delta v)\,\Delta(u_x v_x) = (\Delta + \Delta v)^2 u_x v_x$;

and, generally,

$$\Delta^n(u_x v_x) = (\Delta + \Delta v)^n u_x v_x$$

$$= \Delta^n u_x \,.\, v_x + n\Delta^{n-1} u_x \Delta v_{x+1} + \frac{n(n-1)}{1\,.\,2}\Delta^{n-2} u_x \Delta^2 v_{x+2} + \text{\&c.}$$

which may be also proved inductively by shewing, as in Art. 18, that the coefficients of the developments of $\Delta^n(u_x v_x)$ and $(1+x)^n$, which are identical when $n = 1$, undergo the same changes in passing from n to $n+1$.

Using an accent not to imply that the operation denoted by Δ is altered at all, but merely that Δ' affects v_x only, whilst Δ affects u_x only, the above results are sometimes written,

$$\Delta(u_x v_x) = (\Delta + \Delta' + \Delta\Delta')\, u_x v_x,$$

$$\Delta^2(u_x v_x) = (\Delta + \Delta' + \Delta\Delta')^2 u_x v_x,$$

and, generally, $\quad \Delta^n(u_x v_x) = (\Delta + \Delta' + \Delta\Delta')^n u_x v_x$;

or, if there were more functions w_x, z_x, &c. and we use Δ'', Δ''', &c. to imply that these symbols only affect w_x, z_x, &c. respectively, we have

$$\Delta^n(u_x v_x w_x z_x \ldots)$$

$$= \{(1+\Delta)(1+\Delta')(1+\Delta'')(1+\Delta''')\ldots - 1\}^n u_x v_x w_x z_x \ldots$$

35. To shew that $\Delta^n u_x = (e^{d_x} - 1)^n u_x$, in which the symbols of operation are separated from those of quantity.

By Taylor's Theorem, we have

$$u_{x+n} = u_x + n d_x u_x + \frac{n^2}{1.2} d_x^2 u_x + \&c.,$$

and separating the symbols of operation from those of quantity, we get

$$u_{x+n} = (1 + n d_x + \frac{n^2 d_x^2}{1.2} + \frac{n^3 d_x^3}{1.2.3} + \&c.) u^x$$

$$= e^{n d_x} u_x.$$

Similarly, $u_{x+n-1} = e^{(n-1)d_x} u_x$,

....

$$u_{x+1} = e^{d_x} u_x;$$

$$\therefore \quad \Delta^n u_x = e^{n d_x} u_x - n e^{(n-1)d_x} u_x + \frac{n(n-1)}{1.2} e^{(n-2)d_x} u_x - \&c.,$$

or, again separating the symbols of operation from those of quantity,

$$\Delta^n u_x = (e^{n d_x} - n e^{(n-1)d_x} + \frac{n(n-1)}{1.2} e^{(n-2)d_x} - \&c.) u_x$$

$$= (e^{d_x} - 1)^n u_x,$$

a celebrated theorem first given by Lagrange.

36. To find a general expression for $\Delta^n u_x$ in terms of u_x and its differential coefficients.

The development of the second member of the equation

$$\Delta^n u_x = (e^{d_x} - 1)^n u_x$$

will consist of a series of terms of the form

$$(A_0 + A_1 d_x + \ldots + A_m d_x^m + \ldots)\, u_x,$$

$$\text{or} \quad A_0 u_x + A_1 d_x u_x + \ldots + A_m d_x^m u_x + \ldots,$$

and A_m is evidently the coefficient of t^m in the expansion of $(e^t - 1)^n$.

$$\text{Now} \quad (e^t - 1)^n = e^{nt} - n e^{(n-1)t} + \frac{n(n-1)}{1 \cdot 2} e^{(n-2)t} - \&c.,$$

and the coefficients of t^m in the developments of e^{nt}, $e^{(n-1)t}$, &c. are respectively

$$\frac{n^m}{\lfloor m}, \quad \frac{(n-1)^m}{\lfloor m}, \quad \&c.;$$

$$\therefore \quad A_m = \frac{1}{\lfloor m} \left\{ n^m - n(n-1)^m + \frac{n(n-1)}{1 \cdot 2} (n-2)^m - \&c. \right\}$$

$$= \frac{\Delta^n 0^m}{\lfloor m} \text{ (Art. 20).}$$

Now so long as $m < n$, this vanishes; and when $n = m$, $\Delta^n 0^m = \lfloor n$;

$$\therefore \quad \Delta^n u_x = d_x^n u_x + \frac{\Delta^n 0^{n+1}}{\lfloor n+1} d_x^{n+1} u_x + \frac{\Delta^n 0^{n+2}}{\lfloor n+2} d_x^{n+2} u_x + \&c.$$

Ex. Let $u_x = x^m$, then

$$\Delta^n (x^m) = m(m-1) \ldots (m-n+1)\, x^{m-n}$$

$$+ \frac{\Delta^n 0^{n+1}}{\lfloor n+1} \cdot m(m-1) \ldots (m-n)\, x^{m-n-1} + \&c. + \Delta^n 0^m.$$

37. If in Lagrange's Theorem for $\Delta^n u_x$, $n = 1$, we have $\Delta u_x = (e^{d_x} - 1)\, u_x$, or $\Delta = e^{d_x} - 1$, the meaning of which is, that the operation denoted by Δ is equivalent to the series

of operations denoted by $e^{d_x} - 1$. And, generally, the series of operations denoted by $f(\Delta)$ is equivalent to that denoted by $f(e^{d_x} - 1)$. For let $f(\Delta)$ be developed in a series of the form

$$f(\Delta) = A\Delta^{\alpha} + B\Delta^{\beta} + C\Delta^{\gamma} + \&c.,$$

then $f(\Delta)\,u_x = A\Delta^{\alpha}u_x + B\Delta^{\beta}u_x + C\Delta^{\gamma}u_x + \&c.$

$$= A(e^{d_x}-1)^{\alpha}u_x + B(e^{d_x}-1)^{\beta}u_x + C(e^{d_x}-1)^{\gamma}u_x + \&c.,$$

or, separating the symbols of operation from those of quantity,

$$f(\Delta)\,u_x = \left\{A(e^{d_x}-1)^{\alpha} + B(e^{d_x}-1)^{\beta} + C(e^{d_x}-1)^{\gamma} + \&c.\right\}u_x$$

$$= f(e^{d_x} - 1)\,u_x.$$

38. Suppose $f(\Delta) = (1 + \Delta)^n$,

then $f(e^{d_x} - 1) = (1 + e^{d_x} - 1)^n = e^{nd_x}$;

$\therefore \quad (1 + \Delta)^n u_x = e^{nd_x}u_x = u_{x+n}$, as already proved.

Again, if $f(\Delta) = \{\log(1 + \Delta)\}^n$,

$$f(e^{d_x} - 1) = \{\log(1 + e^{d_x} - 1)\}^n = (\log e^{d_x})^n = d_x^n;$$

$$\therefore \quad \{\log(1 + \Delta)\}^n u_x = d_x^n\,u_x,$$

a formula by which the n^{th} differential coefficient of a function is expressed by its differences.

39. To find the general term of the expansion of $f(e^t)$ in a series ascending by powers of t.

Writing $f(e^t)$ in the form $f\{1 + (e^t - 1)\}$, and expanding by Taylor's Theorem, we find

$$f(e^t) = f(1) + f'(1)(e^t - 1) + \frac{1}{1.2}f''(1)(e^t - 1)^2 + \dots$$

$$+ \frac{1}{\lfloor n}f^{(n)}(1)(e^t - 1)^n + \dots$$

Then taking, as in Art. 36, the coefficient of t^m in each term of the second member, and observing that in $f(1)$ it may be represented by $\dfrac{f(1).0^m}{\lfloor m}$, this quantity being 1 when $m=0$, and zero in all other cases; and that in

$$\frac{1}{\lfloor n} f^{(n)}(1)\,(e^t-1)^n \text{ it is } \frac{1}{\lfloor n} f^{(n)}(1)\,\frac{\Delta^n 0^m}{\lfloor m},$$

we have for the coefficient of t^m in the expansion of $f(e^t)$, the value

$$A_m = \frac{1}{\lfloor m}\left\{f(1)\,0^m + f'(1)\,\Delta 0^m + \frac{1}{1.2} f''(1)\,\Delta^2 0^m + \ldots\right\}$$

$$= \frac{1}{\lfloor m} f(1+\Delta)\,0^m,$$

a remarkable theorem first given by Herschel; for the applications of which, see his Collection of Examples.

SECTION II.

INVERSE METHOD OF DIFFERENCES.

Integration of Explicit Functions.

40. The Inverse Method of Differences has for its object to determine the primitive function from its given difference; or from given relations between it and its differences. We shall begin with the simplest case,

$$\Delta u_x = f(x),$$

in which it is required to determine a function whose difference is given explicitly in terms of the principal variable.

41. Since Δu_x is the difference of $u_x + C$, as well as of u_x, it will be necessary, in passing from the given difference Δu_x to the primitive function, to annex an arbitrary constant C, in order to give the result all the generality of which it is capable. Also C may be a function of x as well as an arbitrary constant, provided its value remains unaltered whilst x changes to $x + 1$. For if C_x denote such a function of x that $C_{x+1} = C_x$, or $\Delta C_x = 0$, we shall have

$$\Delta (u_x + C_x) = \Delta u_x.$$

It is evident that $C_x = \phi (2\lambda\pi x)$ has the property in question, ϕ denoting any trigonometrical function, sine, cosine, &c., and λ any integer.

42. The symbol Σ is used to denote the operation by which we pass from the difference Δu_x to the primitive function; so that

$$\Sigma (\Delta u_x) = u_x + \text{constant}.$$

Also, as the same function admits of successive differences, so a function may be integrated any number of times; the second integral of u_x, or $\Sigma(\Sigma u_x)$, is written $\Sigma^2 u_x$, and the n^{th} integral $\Sigma^n u_x$.

We now proceed to deduce the integrals of various expressions; chiefly, by reversing the processes given in Section I. for finding the differences of functions.

43. It is evident that $\Sigma(u_x + v_x + w_x) = \Sigma u_x + \Sigma v_x + \Sigma w_x$; for if we take the difference of both sides, we get the same result, viz. $u_x + v_x + w_x$. And in the same manner it appears that $\Sigma(au_x) = a\Sigma u_x$.

44. To find the integral of any rational integral function.

Since the difference of a rational integral function is a function of the same kind one dimension lower, it follows that the integral of a function of that description is a similar function one dimension higher; hence, to find the integral of

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n,$$

we may assume it equal to

$$ax^{n+1} + bx^n + \dots + kx + l;$$

then upon taking the difference of each side, and equating the coefficients of like powers of x, there will arise $n+1$ simple equations to determine the $n+1$ quantities $a, b, c, \dots k$; the last term l will remain indeterminate, being in fact the arbitrary constant which must be added to make the integral complete.

Ex. To find $\Sigma(x^4 + 1)$.

Assume $\Sigma(x^4 + 1) = ax^5 + bx^4 + cx^3 + dx^2 + ex$;

$$\therefore\ x^4 + 1 = a(5x^4 + 10x^3 + 10x^2 + 5x + 1) + b(4x^3 + 6x^2 + 4x + 1)$$

$$+ c(3x^2 + 3x + 1) + d(2x + 1) + e;$$

$$\therefore\ 1=5a,\quad 0=10a+4b,\quad 0=10a+6b+3c,\quad 0=5a+4b+3c+2d,$$

$$1=a+b+c+d+e.$$

$$\therefore\ a=\frac{1}{5},\quad b=-\frac{1}{2},\quad c=\frac{1}{3},\quad d=0,\quad e=\frac{29}{30},$$

$$\therefore\ \Sigma(x^4+1)=\frac{x^5}{5}-\frac{x^4}{2}+\frac{x^3}{3}+\frac{29}{30}x+C.$$

45. To find the integral of the product of consecutive terms of an arithmetic progression, we must annex one more factor at the beginning, and divide by the number of factors so increased and by the common difference.

For let $u_x=a+bx$, then we have seen, (Art. 11.) that

$$\Delta u_x u_{x+1}\ldots u_{x+n}=u_{x+1}u_{x+2}\ldots u_{x+n}\,.\,(n+1)\,b,$$

therefore, taking the integrals of both sides, and writing $x-1$ for x, we get

$$\Sigma u_x u_{x+1}\ldots u_{x+n-1}=\frac{u_{x-1}u_x\ldots u_{x+n-1}}{(n+1)\,b}+C,$$

which proves the rule stated above.

Ex. $\Sigma\left(2x+\frac{1}{2}\right)\left(2x+\frac{5}{2}\right)\left(2x+\frac{9}{2}\right)$

$$=\frac{1}{8}\left(2x-\frac{3}{2}\right)\left(2x+\frac{1}{2}\right)\left(2x+\frac{5}{2}\right)\left(2x+\frac{9}{2}\right)+C.$$

If one or more factors be deficient in a factorial of this kind, it may be resolved into others which are complete, as in the following instance;

$$(2x+1)(2x+5)(2x+7)=(2x+3-2)(2x+5)(2x+7)$$

$$=(2x+3)(2x+5)(2x+7)-2(2x+5)(2x+7).$$

46. A rational integral function may often be resolved into factorials of the above form, and in this way its integral more conveniently found, than by the method of Art. 44.

Ex. 1. $x^3 + x^2 = x^2(x+1) = (x-1+1)\,x\,(x+1)$

$$= (x-1)\,x\,(x+1) + x\,(x+1);$$

$$\therefore \quad \Sigma(x^3+x^2) = \frac{(x-2)(x-1)x(x+1)}{4} + \frac{(x-1)x(x+1)}{3} + C.$$

And in general, any quantity of the form

$$ax^n + bx^{n-1} + cx^{n-2} + \text{\&c.}$$

may be resolved into factorials, by the method of indeterminate coefficients; thus, if we assume

$$ax^2 + bx + c = A\,(x+1)(x+2) + B\,(x+1) + C,$$

making $x = -1$, we get $a - b + c = C$;

$$\therefore \quad a\,(x^2-1) + b\,(x+1) = A\,(x+1)(x+2) + B\,(x+1),$$

$$\text{or} \quad a\,(x-1) + b = A\,(x+2) + B;$$

$$\text{make} \quad x = -2, \quad \therefore \quad -3a + b = B;$$

$$\therefore \quad a\,(x-1) + 3a = A\,(x+2), \quad \therefore \quad A = a.$$

In practice, however, it is generally easier to resolve a function by inspection, as in Ex. 1, than by this method, which is theoretically certain.

47. To find the integral of a fraction whose denominator is the product of consecutive terms of an arithmetic progression, and numerator constant, we must efface the last factor, divide by the number of factors remaining and by the common difference, and prefix a negative sign.

For let $u_x = a + bx$, then we have seen (Art. 12.) that

$$\Delta \frac{c}{u_x u_{x+1} \ldots u_{x+n-1}} = - \frac{nbc}{u_x u_{x+1} \ldots u_{x+n}};$$

$\therefore$ taking the integrals of both sides,

$$\Sigma \frac{c}{u_x u_{x+1} \ldots u_{x+n}} = - \frac{c}{nb\, u_x u_{x+1} \ldots u_{x+n-1}} + C,$$

which proves the rule just stated.

48. If the proposed fraction, instead of having its numerator constant, be

$$\frac{Ax^{n-2} + Bx^{n-3} + \ldots + Kx + L}{u_x u_{x+1} \ldots u_{x+n-1}},$$

(the degree of the numerator being at least lower by two units than that of the denominator,) we must reduce the numerator to a series of terms each of which is the product of consecutive factors reckoning from the beginning of the denominator; that is, assume

$$Ax^{n-2} + Bx^{n-3} + \ldots + Kx + L = A' + B'u_x + C'u_x u_{x+1} + \ldots$$
$$+ K'u_x u_{x+1} \ldots u_{x+n-3},$$

then, developing the second member, and equating coefficients of like powers of x, we obtain $n-1$ equations for determining A', B', C', ... K'; and the fraction resolves itself into the following, each of which is integrable,

$$\frac{A'}{u_x \ldots u_{x+n-1}} + \frac{B'}{u_{x+1} \ldots u_{x+n-1}} + \&c. + \frac{K'}{u_{x+n-2} \cdot u_{x+n-1}}.$$

If the degree of the numerator were only lower by one unit than that of the denominator, we should arrive at a term $\frac{L'}{u_{x+n-1}}$, of which we are able to find the integral, only approximately.

Hence also, if any of the factors of the denominator of the fraction in Art. 47 be wanting, they may be supplied by introducing them into the numerator and denominator at the same time; and then the resulting fraction may be treated as in the present Article.

$$\text{Ex.}\quad \frac{1}{x^2-4} = \frac{(x-1)\,x\,(x+1)}{(x-2)\ldots(x+2)}$$

$$= \frac{1}{(x+1)(x+2)} + \frac{3}{x(x+1)(x+2)} + \frac{6}{(x-1)(x+1)(x+2)}$$

$$+ \frac{6}{(x-2)(x-1)\,x\,(x+1)(x+2)},$$

which is got by assuming

$$(x-1)\,x\,(x+1) = a(x-2)(x-1)\,x + b(x-2)(x-1) + c(x-2) + d,$$

and making $x = 2$, 1, 0, successively; taking care to reject the factor common to both sides, after each substitution.

49. To find the integral of a^x.

We have seen (Art. 14.) that $\Delta a^x = (a-1)\,a^x$;

$$\therefore\ \Sigma a^x = \frac{a^x}{a-1} + C.\qquad \text{Also }\ \Sigma^n a^x = \frac{a^x}{(a-1)^n},$$

suppressing the part introduced by the constants, which would be a rational integral function of the $(n-1)^{\text{th}}$ degree.

50. To find the integral of $\log v_x$.

If $u_x = \log(v_1 v_2 v_3 \ldots v_{x-1})$,

then $\Delta u_x = \log(v_1 v_2 \ldots v_x) - \log(v_1 v_2 \ldots v_{x-1}) = \log v_x$;

$$\therefore\ \Sigma \log v_x = u_x + \log C = \log(C.v_1 v_2 \ldots v_{x-1}) = \log CPv_{x-1},$$

using Pv_x to denote the product of all the successive values of the function v_x, from some fixed term v_1, (or more generally v_n, n being independent of x) to v_x inclusive.

51. To find the integrals of $\cos x\theta$, $\sin x\theta$.

Since $\Delta \cos x\theta = -2\sin\frac{\theta}{2}\sin(x+\frac{1}{2})\theta$,

$$\therefore\ \Delta\cos(x-\tfrac{1}{2})\theta = -2\sin\frac{\theta}{2}\sin x\theta;$$

$$\therefore\ \Sigma\sin x\theta = -\frac{\cos(x-\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} + C.$$

Also, since

$$\Delta^n \sin(x\theta+a) = \left(2\sin\frac{\theta}{2}\right)^n \sin\left\{x\theta + a + \frac{n}{2}(\pi+\theta)\right\}, \text{ (Art. 16.)},$$

integrating both sides n times, and replacing a by $a - \frac{n}{2}(\pi+\theta)$, we get

$$\Sigma^n \sin(x\theta+a) = \frac{\sin\left\{x\theta + a - \frac{n}{2}(\pi+\theta)\right\}}{\left(2\sin\frac{\theta}{2}\right)^n}.$$

Again, $\Delta\sin x\theta = 2\sin\frac{\theta}{2}\cos(x+\frac{1}{2})\theta$,

$$\therefore\ \Delta\sin(x-\tfrac{1}{2})\theta = 2\sin\frac{\theta}{2}\cos x\theta;$$

$$\therefore\ \Sigma\cos x\theta = \frac{\sin(x-\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} + C,$$

$$\text{and}\ \Sigma^n\cos(x\theta+a) = \frac{\cos\left\{x\theta + a - \frac{n}{2}(\pi+\theta)\right\}}{\left(2\sin\frac{\theta}{2}\right)^n}.$$

52. The preceding expressions may also be integrated by substituting for them their exponential values; as in the fol ving instance.

$$\Sigma a^x \cos x\theta = \tfrac{1}{2}\Sigma\left(a^x e^{x\theta\sqrt{-1}} + a^x e^{-x\theta\sqrt{-1}}\right)$$

$$= \frac{1}{2}\frac{a^x e^{x\theta\sqrt{-1}}}{ae^{\theta\sqrt{-1}} - 1} + \frac{1}{2}\frac{a^x e^{-x\theta\sqrt{-1}}}{ae^{-\theta\sqrt{-1}} - 1}$$

$$= \frac{a^x}{2}\,\frac{ae^{(x-1)\theta\sqrt{-1}} + ae^{-(x-1)\theta\sqrt{-1}} - e^{x\theta\sqrt{-1}} - e^{-x\theta\sqrt{-1}}}{a^2 - a\left(e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}\right) + 1}$$

$$= a^x\,\frac{a\cos(x-1)\theta - \cos x\theta}{a^2 - 2a\cos\theta + 1} + C.$$

Hence, putting $a^2 - 2a\cos\theta + 1 = c$, and denoting

$a^x\cos x\theta$ by u_x, we have $c\Sigma u_x = a^2 u_{x-1} - u_x$,

$$c^2\Sigma^2 u_x = a^4 u_{x-2} - 2a^2 u_{x-1} + u_x,$$

and generally,

$$c^n\Sigma^n u_x = a^{2n}u_{x-n} - na^{2n-2}u_{x-n+1} + \frac{n(n-1)}{1.2}a^{2n-4}u_{x-n+2} - \&c. \pm u_x.$$

Exactly the same formulæ hold for $u_x = a^x\sin x\theta$.

In the same manner the integrals of $a^x(\sin x\theta)^m$, $a^x(\cos x\theta)^m$ may be obtained.

53. To find the integral of $\dfrac{1}{\cos x\theta\cos(x+1)\theta}$.

Since $\Delta\tan x\theta = \dfrac{\sin\theta}{\cos x\theta\cos(x+1)\theta}$, (Art. 17.)

we have $\Sigma\dfrac{1}{\cos x\theta\cos(x+1)\theta} = \dfrac{\tan x\theta}{\sin\theta} + C.$

54. To find the integral of $\tan^{-1}\dfrac{1}{p+qx+rx^2}$.

Since $\Delta\tan^{-1}(a+bx)=\tan^{-1}\dfrac{b}{1+(a+bx)(a+bx+b)}$, (Art. 17.)

$$\text{we may assume } \Sigma\tan^{-1}\frac{1}{p+qx+rx^2}=\tan^{-1}(a+bx),$$

and take the difference of both sides; then if the proposed function is capable of being integrated, the indeterminate coefficients a and b will become known.

55. The integral of $a^x u_x u_{x+1} \ldots u_{x+n-1}$, where $u_x=pa^x+q$, may be determined by assuming it equal to the same expression (only with another factor at the beginning instead of a^x) multiplied by an indeterminate coefficient; for

$$\Delta u_{x-1}u_x \ldots u_{x+n-1}=u_x u_{x+1} \ldots u_{x+n-1}(u_{x+n}-u_{x-1})$$

$$=a^x u_x u_{x+1} \ldots u_{x+n-1}\cdot p(a^n-a^{-1});$$

$$\therefore\ \Sigma a^x u_x \ldots u_{x+n-1}=\frac{a}{p(a^{n+1}-1)}\cdot u_{x-1}u_x \ldots u_{x+n-1}.$$

Similarly, to find $\Sigma\dfrac{a^x}{u_x u_{x+1} \ldots u_{x+n}}$, for the assumption we must efface the last factor in the denominator, and write instead of a^x an indeterminate coefficient; for

$$\Delta\frac{1}{u_x u_{x+1} \ldots u_{x+n-1}}=-\frac{u_{x+n}-u_x}{u_x u_{x+1} \ldots u_{x+n}}=-\frac{pa^x(a^n-1)}{u_x u_{x+1} \ldots u_{x+n}};$$

$$\therefore\ \Sigma\frac{a^x}{u_x u_{x+1} \ldots u_{x+n}}=-\frac{1}{p(a^n-1)}\cdot\frac{1}{u_x u_{x+1} \ldots u_{x+n-1}}.$$

56. In like manner, if $u_x = a + bx$, expressions of the forms

$$\frac{(p+qx)\,t^x}{u_x u_{x+1} \ldots u_{x+n-1}}, \quad \frac{(p+qx+rx^2)\,t^x}{u_x u_{x+1} \ldots u_{x+n-1}},$$

can sometimes be integrated, by assuming their integrals equal to expressions of the same form, except that the last factor in the denominator is effaced, and the polynomial in the numerator is replaced by another one dimension lower with indeterminate coefficients. It is of course only when a certain equation of condition between the quantities a, b, p, q, t is satisfied, that this method succeeds.

Ex. 1. Let $\Sigma \dfrac{x+1}{(2x-1)(2x+1)\,3^x} = \dfrac{A\left(\frac{1}{3}\right)^x}{2x-1}$;

$$\therefore \quad \frac{x+1}{(2x-1)(2x+1)\,3^x} = A\left(\tfrac{1}{3}\right)^{x+1}\left(\frac{1}{2x+1} - \frac{3}{2x-1}\right)$$

$$= A\left(\tfrac{1}{3}\right)^{x+1}\frac{-4(x+1)}{(2x-1)(2x+1)};$$

$$\therefore \quad A = -\tfrac{3}{4}, \quad \text{and} \quad \Sigma u_x = C - \tfrac{3}{4}\,\frac{1}{(2x-1)\,3^x}.$$

Ex. 2. Let $\Sigma \dfrac{x^2+6x+12}{x(x+1)(x+2)\,2^x} = \dfrac{A+Bx}{x(x+1)\,2^x}$,

then $A = -6$, $B = -2$, and $\Sigma u_x = C - \dfrac{x+3}{x(x+1)\,2^{x-1}}$.

57. Since $\Delta(u_x v_x) = u_x \Delta v_x + v_{x+1}\Delta u_x$, we have

$$\Sigma(u_x \Delta v_x) = u_x v_x - \Sigma(v_{x+1}\Delta u_x),$$

the formula for integration by parts, corresponding to the formula

$$\int_x u\,d_x v = uv - \int_x v\,d_x u.$$

58. Change v_x into Σv_x, and consequently Δv_x into v_x,

then $\Sigma(u_x v_x) = u_x \Sigma v_x - \Sigma(\Delta u_x \Sigma v_{x+1})$;

hence, by successive substitutions, we get

$$\Sigma(\Delta u_x \Sigma v_{x+1}) = \Delta u_x \Sigma^2 v_{x+1} - \Sigma(\Delta^2 u_x \Sigma^2 v_{x+2}),$$

$$\Sigma(\Delta^2 u_x \Sigma^2 v_{x+2}) = \Delta^2 u_x \Sigma^3 v_{x+2} - \Sigma(\Delta^3 u_x \Sigma^3 v_{x+3}),$$

$$\dots\dots\dots\dots\dots\dots$$

$$\Sigma(\Delta^n u_x \Sigma^n v_{x+n}) = \Delta^n u_x \Sigma^{n+1} v_{x+n} - \Sigma(\Delta^{n+1} u_x \Sigma^{n+1} v_{x+n+1});$$

$$\therefore \quad \Sigma(u_x v_x) = u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_x \Sigma^3 v_{x+2}$$
$$- \dots \pm \Delta^n u_x \Sigma^{n+1} v_{x+n} \mp \Sigma(\Delta^{n+1} u_x \Sigma^{n+1} v_{x+n+1}).$$

Obs. In order to be more easily remembered, this formula may be written

$$\Sigma(u_x v_x) = (1 + \Delta \Sigma v)^{-1} u_x \Sigma v_x,$$

where Δ affects u_x only, and Σv affects Σv_x only, and implies an operation such that

$$(\Sigma v)^r \Sigma v_x = \Sigma^{r+1} v_{x+r}.$$

Hence,

$$\Sigma^2(u_x v_x) = (1 + \Delta \Sigma v)^{-1} \Sigma(u_x \Sigma v_x) = (1 + \Delta \Sigma v)^{-2} u_x \Sigma^2 v_x;$$

and, generally,

$$\Sigma^n(u_x v_x) = (1 + \Delta \Sigma v)^{-n} u_x \Sigma^n v_x = u_x \Sigma^n v_x - n \Delta u_x \Sigma^{n+1} v_{x+1}$$
$$+ \frac{n(n+1)}{1 . 2} \Delta^2 u_x \Sigma^{n+2} v_{x+2} - \&c., \quad (1)$$

which may be also proved inductively, by shewing that the coefficients of the developments of $\Sigma^n(u_x v_x)$ and $(1+x)^{-n}$, which are identical when $n = 1$, undergo the same changes in passing from n to $n-1$. This appears by differencing both sides of (1).

59. The above formula always enables us to find the integrals of functions made up of two factors, one of which leads to zero, as the value of one of its successive differences, and the other admits of successive integrations. Suppose, for example, that u_x is a rational integral function of the n^{th} degree, and that $v_x = a^x$; then

$$\Sigma v_x = \frac{a^x}{a-1}, \quad \Sigma^2 v_{x+1} = \frac{a^{x+1}}{(a-1)^2}, \ \&c.; \ \Delta^n u_x = \text{const}, \ \Delta^{n+1} u_x = 0;$$

$$\therefore \quad \Sigma(u_x a^x) = \frac{u^x a^x}{a-1} - \frac{\Delta u_x a^{x+1}}{(a-1)^2} + \frac{\Delta^2 u_x a^{x+2}}{(a-1)^3} - \ldots \pm \frac{\Delta^n u_x a^{x+n}}{(a-1)^{n+1}} + C.$$

Again, suppose u_x to be a rational integral function of the n^{th} degree, and $v_x = \cos x\theta$; then

$$\Sigma^n v_x = \frac{\cos\left\{x\theta - \frac{n}{2}(\pi + \theta)\right\}}{\left(2 \sin \frac{\theta}{2}\right)^n} \text{ (Art. 51.)};$$

$$\therefore \quad \Sigma(u_x \cos x\theta) = \frac{u_x \cos\left\{x\theta - \frac{1}{2}(\pi + \theta)\right\}}{2 \sin \frac{\theta}{2}}$$

$$- \Delta u_x \frac{\cos\left\{(x+1)\theta - (\pi + \theta)\right\}}{\left(2 \sin \frac{\theta}{2}\right)^2} + \Delta^2 u_x \frac{\cos\left\{(x+2)\theta - \frac{3}{2}(\pi + \theta)\right\}}{\left(2 \sin \frac{\theta}{2}\right)^3}$$

$$- \text{\&c.},$$

the series terminating with $\Delta^n u_x$. Similarly, if $v_x = a^x \cos x\theta$; or if $v_x = a^x \cos^m x\theta \sin^n x\theta$, since the product $\cos^m x\theta \sin^n x\theta$ may be replaced by simple dimensions of sines and cosines of multiples of $x\theta$; and it will be noticed that the fraction of Art. 48 may be brought under this case.

60. Since the performance of the operation Σ upon any series of terms $A\Delta^m u_x + B\Delta^n u_x + \ldots$, reduces it to

$$A\Delta^{m-1} u_x + B\Delta^{n-1} u_x + \ldots;$$

it appears that prefixing Σ to $\{A\Delta^m + B\Delta^n + \ldots\} u_x$ has the same effect as prefixing Δ^{-1}; in other words, Σ is equivalent to Δ^{-1}. And in like manner, since integrating $\Delta^m u_x$ n times, reduces it to $\Delta^{m-n} u_x$, Σ^n must be equivalent to Δ^{-n}.

The same reasoning is applicable to the symbols $\int_x^n$, d_x^{-n}; whenever therefore, in separating the symbols of operation from those of quantity, as in the expression $F(\Delta) u_x$, $f(d_x) u_x$, terms containing negative powers of Δ and d_x occur, they

must be understood to be replaced by the corresponding positive powers of Σ and $\int_x$. This being premised, we proceed to investigate a general series for Σu_x; preparatory to which the following propositions must be proved.

61. To determine the generating functions of

$$\Sigma^n u_x, \quad d_x^n u_x, \quad \int_x^n u_x,$$

from that of u_x.

By virtue of the relations

$$G(\Delta u_x) = \left(\frac{1}{t} - 1\right) Gu_x,$$

$$G(\Delta^n u_x) = \left(\frac{1}{t} - 1\right)^n Gu_x,$$

we have $$\left(\frac{1}{t} - 1\right) G(\Sigma u_x) = G(\Delta \Sigma u_x) = Gu_x;$$

$$\therefore \quad G(\Sigma u_x) = \left(\frac{1}{t} - 1\right)^{-1} Gu_x,$$

and $$G(\Sigma^n u_x) = \left(\frac{1}{t} - 1\right)^{-n} Gu_x.$$

62. Again, since

$$Gu_x = \ldots + u_x t^x + \ldots + u_{x+h} t^{x+h} + \ldots;$$

$$\therefore \quad \ldots + (u_{x+h} - u_x) t^x + \ldots = \left(\frac{1}{t^h} - 1\right) Gu_x$$

$$= \left\{1 - h \log t + \frac{h^2}{1.2} (\log t)^2 - \&c. - 1\right\} Gu_x;$$

therefore, dividing both sides by h and then making $h = 0$, which we are at liberty to do since h is indeterminate,

$$\ldots + \left\{\frac{u_{x+h} - u_x}{h}\right\}_{h=0} t^x + \ldots = -\log t\, Gu_x = \log\frac{1}{t}\,.\,Gu_x,$$

$$\text{or} \quad \ldots + d_x u_x\,.\,t^x + \ldots = \log\frac{1}{t}\,.\,Gu_x;$$

$$\therefore \quad G(d_x u_x) = \log\frac{1}{t}\,.\,Gu_x,$$

$$\text{and} \quad G(d_x^n u_x) = \left(\log\frac{1}{t}\right)^n Gu_x.$$

Again, $\log\frac{1}{t}\,G(\int_x u_x) = G(d_x \int_x u_x) = Gu_x;$

$$\therefore \quad G(\int_x u_x) = \left(\log\frac{1}{t}\right)^{-1} Gu_x, \qquad \text{and} \qquad G(\int_x^n u_x) = \left(\log\frac{1}{t}\right)^{-n} Gu_x.$$

63. To investigate a general series for Σu_x, involving only $\int_x u_x$, u_x, and the differential coefficients of u_x.

$$G(\Sigma u_x) = \left(\frac{1}{t} - 1\right)^{-1} Gu_x = \left(e^{\log\frac{1}{t}} - 1\right)^{-1} Gu_x$$

$$= \frac{Gu_x}{\log\frac{1}{t}}\left\{\frac{\log\frac{1}{t}}{e^{\log\frac{1}{t}} - 1}\right\}$$

$$= \left(\log\frac{1}{t}\right)^{-1} Gu_x \left\{1 - \tfrac{1}{2}\log\frac{1}{t} + \frac{B_1}{1\,.\,2}\left(\log\frac{1}{t}\right)^2 - \&c.\right.$$

$$\left. + (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}\left(\log\frac{1}{t}\right)^{2n} + \ldots\right\},$$

assuming, as will be proved in the next Art., that $\dfrac{v}{e^v - 1}$ can be expanded in a series of the same form as that within brackets, and denoting by

$$\frac{B_1}{1\,.\,2}, \quad \frac{-B_3}{1\,.\,2\,.\,3\,.\,4}, \quad \&c., \quad (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}$$

the coefficients of v^2, v^4, ... v^{2n} in that expansion. Hence

$$G(\Sigma u_x) = G(\textstyle\int_x u_x) - \frac{1}{2} G u_x + \frac{B_1}{1 \cdot 2} G(d_x u_x) - \&c.$$

$$= G\left\{\textstyle\int_x u_x - \frac{1}{2} u_x + \frac{B_1}{1 \cdot 2} d_x u_x - \frac{B_3}{\underline{|4}} d_x^3 u_x + \&c.\right\};$$

$$\therefore\ \Sigma u_x = \textstyle\int_x u_x - \frac{1}{2} u_x + \frac{B_1}{1 \cdot 2} d_x u_x - \frac{B_3}{\underline{|4}} d_x^3 u_x + \dots$$

$$+ (-1)^{n+1} \frac{B_{2n-1}}{\underline{|2n}} d_x^{2n-1} u_x + \dots$$

Ex. 1. Let $u_x = x^m$;

$$\therefore\ \Sigma x^m = \frac{x^{m+1}}{m+1} - \frac{1}{2} x^m + \frac{1}{2} B_1 m x^{m-1}$$

$$- \frac{1}{4} B_3 \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^{m-3} + \&c. + C.$$

Hence $\Sigma x^4 = \dfrac{x^5}{5} - \dfrac{x^4}{2} + \dfrac{x^3}{3} - \dfrac{x}{30} + C$;

$$\Sigma x^5 = \frac{x^6}{6} - \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12} + C.$$

Ex. 2. Let $u_x = \dfrac{1}{a+bx}$;

$$\therefore\ \Sigma \frac{1}{a+bx} = \frac{1}{b} \log(a+bx) - \frac{1}{2(a+bx)}$$

$$- \frac{B_1 b}{2(a+bx)^2} + \frac{B_3 b^3}{4(a+bx)^4} - \&c. + C.$$

Numbers of Bernouilli.

64. The numbers B_1, B_3,... which are required in the general value of Σu_x in the preceding Art., are called the numbers of Bernouilli, and are of great importance in the theory of Series. They are defined by the equation

$$\frac{t}{e^t-1} = 1 - \frac{1}{2}t + \frac{B_1}{1.2}t^2 - \frac{B_3}{1.2.3.4}t^4 + \ldots$$
$$+ (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}t^{2n} \pm \&c.,$$

and their values may be computed in the following manner.

The development of $\dfrac{t}{e^t-1}$ can involve no odd powers of t above the first. For if $\phi(t) = \dfrac{t}{e^t-1}$, we have

$$\phi(-t) = \frac{-t}{e^{-t}-1} = \frac{te^t}{e^t-1},$$

$$\therefore\ \phi(t) - \phi(-t) = -t, \quad \text{and } d_t^2\phi(t) - d_t^2\phi(-t) = 0,$$
$$\text{or } d_t^2\phi(t) = d_t^2\phi(-t).$$

Hence, when the development is twice differentiated, it will not be altered by changing t into $(-t)$, and therefore contains only even powers of t; therefore $\phi(t)$ can only contain one odd power of t viz. the first. Also since

$$\frac{t}{e^t-1} = \frac{1}{1 + \frac{t}{2} + \frac{t^2}{1.2.3} + \&c.},$$

we may assume for $\phi(t)$ the form of development given above, viz.

$$\frac{t}{e^t-1} = 1 - \frac{1}{2}t + \frac{B_1}{1.2}t^2 - \frac{B_3}{\lfloor 4}t^4 + \ldots + (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}t^{2n} \pm \&c.$$

Now $$\frac{t}{e^t-1}=\frac{\log\{1+(e^t-1)\}}{e^t-1}=1-\tfrac{1}{2}(e^t-1)+\tfrac{1}{3}(e^t-1)^2-\ldots$$
$$+\frac{1}{2n+1}(e^t-1)^{2n}\mp\&c.,$$

and if A_{2n} be the coefficient of t^{2n} in this development of the second member,

$$A_{2n}=-\tfrac{1}{2}\frac{1^{2n}}{\lfloor 2n}+\tfrac{1}{3}\cdot\frac{2^{2n}-2\,.\,1^{2n}}{\lfloor 2n}-\&c.$$
$$+\frac{1}{2n+1}\cdot\frac{(2n)^{2n}-2n(2n-1)^{2n}+\ldots}{\lfloor 2n}\mp\&c.$$
$$=\frac{1}{\lfloor 2n}\left\{-\tfrac{1}{2}\Delta 0^{2n}+\tfrac{1}{3}\,.\,\Delta^2 0^{2n}-\&c.+\frac{1}{2n+1}\Delta^{2n}0^{2n}\right\};$$

all the terms after $\Delta^{2n}0^{2n}$ vanishing, since $\Delta^m 0^{2n}$ is zero when $m>2n$.

Hence,

$$B_{2n-1}=(-1)^{n+1}\left\{-\tfrac{1}{2}\Delta 0^{2n}+\tfrac{1}{3}\Delta^2 0^{2n}-\&c.\ldots+\frac{1}{2n+1}\Delta^{2n}0^{2n}\right\}.$$

By this formula B_1, B_3, B_5, &c. may be readily computed, supposing the numbers comprised in the form $\Delta^m 0^{2n}$ to be known; we find

$$B_1=\frac{1}{6},\quad B_3=\frac{1}{30},\quad B_5=\frac{1}{42},\quad B_7=\frac{1}{30},\quad B_9=\frac{5}{66},\ \&c.$$

65. Also, since

$$\Delta^m 0^{2n}=m^{2n}-m(m-1)^{2n}+\frac{m(m-1)}{1\,.\,2}(m-2)^{2n}-\&c.\ \text{(Art. 20.)}$$

we may, if we please, eliminate the numbers $\Delta^m 0^{2n}$ from the expression for B_{2n-1}; and we find

$$(-1)^{n+1}B_{2n-1}=-\tfrac{1}{2}+\tfrac{1}{3}(2^{2n}-2)-\tfrac{1}{4}(3^{2n}-3\,.\,2^{2n}+3)+\&c.+\frac{\lfloor 2n}{2n+1}.$$

Besides serving to express the general value of Σu_x, the numbers of Bernouilli have various other uses, of which we shall now give one or two of the most remarkable.

66. To find the general terms of the expansions of $\cot\theta$ and $\tan\theta$ in powers of θ.

$$\cot\theta = \sqrt{-1}\left(1+\frac{2}{e^{2\theta\sqrt{-1}}-1}\right) = \sqrt{-1}+\frac{1}{\theta}\cdot\frac{2\theta\sqrt{-1}}{e^{2\theta\sqrt{-1}}-1}.$$

Now the general term of the expansion of

$$\frac{2\theta\sqrt{-1}}{e^{2\theta\sqrt{-1}}-1} \text{ is } (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}(2\theta\sqrt{-1})^{2n},$$

$$\text{or } -\frac{2^{2n}B_{2n-1}}{\lfloor 2n}\theta^{2n};$$

$\therefore$ the general term of the expansion of $\cot\theta$ is

$$-\frac{2^{2n}B_{2n-1}\theta^{2n-1}}{\lfloor 2n},$$

$$\text{and} \quad \cot\theta = \frac{1}{\theta}-\frac{2^2}{1.2}B_1\theta-\frac{2^4 B_3}{1.2.3.4}\theta^3-\&c.$$

Also since $\tan\theta=\cot\theta-2\cot 2\theta$,

the general term of the series for $\tan\theta$ is

$$-\frac{2^{2n}B_{2n-1}\theta^{2n-1}}{\lfloor 2n}+2.\frac{2^{2n}B_{2n-1}(2\theta)^{2n-1}}{\lfloor 2n}$$

$$=\frac{2^{2n}(2^{2n}-1)B_{2n-1}}{\lfloor 2n}\theta^{2n-1},$$

$$\therefore \tan\theta = \frac{4.3}{1.2}B_1\theta+\frac{16.15}{1.2.3.4}B_3\theta^3+\frac{2^6(2^6-1)}{1.2\ldots 6}B_5\theta^5+\&c.$$

Hence, by differentiating the above expressions for $\cot\theta$ and $\tan\theta$, we may deduce the general terms of the expansions of $\cot^2\theta$ and $\tan^2\theta$; and by integrating them, the general terms of the expansions of $\log\sin\theta$, $\log\cos\theta$.

67. To find the sum of the series

$$\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\text{ (to }\infty\text{)},$$

or the value of ${}^1S^{\infty}\dfrac{1}{x^{2n}}$.

Since $$\sin\theta=\theta\left\{1-\left(\frac{\theta}{\pi}\right)^2\right\}\left\{1-\left(\frac{\theta}{2\pi}\right)^2\right\}\left\{1-\left(\frac{\theta}{3\pi}\right)^2\right\}\dots$$

changing θ into $\pi\theta$, we get

$$\sin\pi\theta=\pi\theta\left\{1-\left(\frac{\theta}{1}\right)^2\right\}\left\{1-\left(\frac{\theta}{2}\right)^2\right\}\left\{1-\left(\frac{\theta}{3}\right)^2\right\}\dots$$

therefore, differentiating the logarithm of each member,

$$\pi\cot\pi\theta=\frac{1}{\theta}-\frac{2\theta}{1^2}\left\{1-\left(\frac{\theta}{1}\right)^2\right\}^{-1}-\frac{2\theta}{2^2}\left\{1-\left(\frac{\theta}{2}\right)^2\right\}^{-1}$$

$$-\frac{2\theta}{3^2}\left\{1-\left(\frac{\theta}{3}\right)^2\right\}^{-1}-\&c.$$

$$=\frac{1}{\theta}-2\theta\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)-2\theta^3\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right)-\&c.$$

$$-2\theta^{2n-1}\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\right)-\dots$$

But the coefficient of θ^{2n-1} in the expansion of $\pi\cot\pi\theta$ is

$$-\frac{2^{2n}B_{2n-1}\pi^{2n}}{\lfloor 2n},$$

$$\therefore \quad {}_1S^{\infty}\frac{1}{x^{2n}} = \frac{2^{2n-1}B_{2n-1}\pi^{2n}}{1.2.3\ldots 2n}.$$

Hence $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$, $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}$.

Obs. Calling S_{2n} the sum of this series, we have

$$\frac{S_{2n}}{S_{2n+2}} = \frac{(2n+1)(2n+2)}{4\pi^2}\,\frac{B_{2n-1}}{B_{2n+1}},$$

Now suppose n very great, then $\frac{B_{2n+1}}{B_{2n-1}} = \frac{n^2}{\pi^2}$, which proves the divergency of the series formed by the numbers of Bernouilli; these numbers increase very rapidly, beginning with B_{13}.

68. To find the sum of the series

$$\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\ldots\text{ (to } \infty\text{)},$$

or the value of ${}_0S^{\infty}\frac{1}{(2x+1)^{2n}}$.

Since $\cos\theta = \left\{1-\left(\frac{2\theta}{\pi}\right)^2\right\}\left\{1-\left(\frac{2\theta}{3\pi}\right)^2\right\}\left\{1-\left(\frac{2\theta}{5\pi}\right)^2\right\}\ldots$

changing θ into $\frac{\pi\theta}{2}$, we get

$$\cos\frac{\pi\theta}{2} = \left\{1-\left(\frac{\theta}{1}\right)^2\right\}\left\{1-\left(\frac{\theta}{3}\right)^2\right\}\left\{1-\left(\frac{\theta}{5}\right)^2\right\}\ldots$$

therefore, differentiating the logarithm of each member,

$$\frac{\pi}{2}\tan\frac{\pi\theta}{2} = \frac{2\theta}{2}\left\{1-\left(\frac{\theta}{1}\right)^2\right\}^{-1}+\frac{2\theta}{3^2}\left\{1-\left(\frac{\theta}{3}\right)^2\right\}^{-1}$$

$$+\frac{2\theta}{5^2}\left\{1-\left(\frac{\theta}{5}\right)^2\right\}^{-1}+\&c.;$$

and equating the coefficients of θ^{2n-1} in each member, which are respectively,

$$\left(\frac{\pi}{2}\right)^{2n} B_{2n-1} \frac{2^{2n}(2^{2n}-1)}{\lfloor 2n}, \quad \text{and} \quad 2\,{}^0S^{\infty} \frac{1}{(2x+1)^{2n}},$$

$$\text{we get} \quad {}^0S^{\infty} \frac{1}{(2x+1)^{2n}} = \tfrac{1}{2} B_{2n-1} \frac{\pi^{2n}(2^{2n}-1)}{1.2.3\ldots 2n}.$$

$$\text{Hence} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c. = \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{3\pi^2}{1.2} = \frac{\pi^2}{8}.$$

69. Also $\frac{1}{1^{2n}} - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \ldots$

$$= \frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \ldots - \frac{1}{2^{2n}}\left(\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots\right)$$

$$= \frac{1}{2} \frac{2^{2n}-1}{\lfloor 2n} \pi^{2n} B_{2n-1} - \frac{1}{2^{2n}} \frac{2^{2n-1}\pi^{2n}}{\lfloor 2n} B_{2n-1}$$

$$= \frac{(2^{2n-1}-1)\,\pi^{2n} B_{2n-1}}{\lfloor 2n}.$$

70. Since $\log(1+e^{-z}) = e^{-z} - \frac{1}{2}e^{-2z} + \frac{1}{3}e^{-3z} - \&c.,$ integrating this $2n-1$ times between the limits $z=0$, $z=\infty$, we find for result the series just summed;

$$\therefore \quad ({}^0\!\int_z^{\infty})^{2n-1} \log(1+e^{-z}) = \frac{(2^{2n-1}-1)\,\pi^{2n} B_{2n-1}}{\lfloor 2n}.$$

71. To find an approximate value of

$\Gamma(x+1) = 1.2.3\ldots x$, when x is very large.

Making $u_x = \log x$ in the formula

$$\Sigma u_x = \int_x u_x - \tfrac{1}{2} u_x + \frac{B_1}{1.2} d_x u_x - \&c.$$

we find, (Art. 50.) adding $\log x$ to both sides,

$$\log\{1 . 2 . 3 \ldots (x-1)\,x\} = C + x\log x - x - \tfrac{1}{2}\log x$$
$$+ \frac{B_1}{1 . 2}\frac{1}{x} - \frac{B_3}{3 . 4}\frac{1}{x^3} + \ldots + \log x$$
$$= C + (x + \tfrac{1}{2})\log x - x + \log(1+h),$$

putting $\log(1+h) = \dfrac{B_1}{1 . 2x} - \dfrac{B_3}{3 . 4x^3} + \dfrac{B_5}{5 . 6x^5} - \&c.,$

so that h is a quantity continually approaching to zero as x increases.

Now to determine C, suppose x very large so that h may be neglected, and change x into $2x$, then

$$\log(1 . 2 . 3 \ldots 2x) = C + (2x + \tfrac{1}{2})\log 2x - 2x$$
$$= C + (2x + \tfrac{1}{2})(\log x + \log 2) - 2x,$$

and $\log(2 . 4 . 6 \ldots 2x) = \log(2^x . 1 . 2 . 3 \ldots x)$

$$= x\log 2 + C + (x + \tfrac{1}{2})\log x - x;$$

$$\therefore \log\{1 . 3 . 5 \ldots (2x-1)\} = x\log x + (x + \tfrac{1}{2})\log 2 - x;$$

$$\therefore \log\frac{2 . 4 . 6 \ldots 2x}{1 . 3 . 5 \ldots (2x-1)} = C + \tfrac{1}{2}\log x - \tfrac{1}{2}\log 2$$
$$= C + \tfrac{1}{2}\log 2x - \log 2;$$

$$\therefore 2C - 2\log 2 = 2\log\frac{2 . 4 . 6 \ldots (2x)}{1 . 3 . 5 \ldots (2x-1)} - \log 2x$$

$$= \log\frac{2 . 2 . 4 . 4 . 6 . 6 \ldots (2x-2)\,2x}{1 . 3 . 3 . 5 . 5 . 7 \ldots (2x-1)(2x-1)}$$

$$= \log\frac{\pi}{2},$$ by Wallis's Theorem,

since x is indefinitely large;

$$\therefore \quad C = \tfrac{1}{2} \log 2\pi;$$

$$\therefore \quad \log(1.2.3\ldots x) = \tfrac{1}{2}\log 2\pi + (x + \tfrac{1}{2})\log x - x + \log(1+h)$$

$$= \log\sqrt{2\pi x} + \log\left(\frac{x}{e}\right)^x + \log(1+h);$$

$$\therefore \quad 1.2.3\ldots x = \sqrt{2\pi x}\,.\left(\frac{x}{e}\right)^x.(1+h),$$

where h is to be calculated from the series

$$\log(1+h) = \frac{B_1}{1.2}\cdot\frac{1}{x} - \frac{B_3}{3.4}\cdot\frac{1}{x^3} + \frac{B_5}{5.6}\cdot\frac{1}{x^5} - \&c.,$$

and in general it will suffice to take the first term only, which gives

$$1.2.3\ldots x = \sqrt{2\pi x}\, x^x e^{-x+\frac{1}{12x}}.$$

Obs. The preceding series, even for large values of x, becomes divergent after a certain number of terms; this will happen after n terms if

$$\frac{B_{2n+1}}{(2n+1)(2n+2)}\cdot\frac{1}{x^{2n+1}} > \frac{B_{2n-1}}{(2n-1)\,2n}\,\frac{1}{x^{2n-1}},$$

$$\text{or} \quad \frac{B_{2n+1}}{B_{2n-1}} > \frac{(2n+1)(2n+2)}{(2n-1)\,2n}\,x^2,$$

but the first member of this inequality never exceeds

$$\frac{(2n+1)(2n+2)}{4\pi^2}, \quad \text{(Art. 67.)}$$

$$\therefore \quad (2n-1)\,2n > 4\pi^2 x^2, \quad \text{or} \quad n > \pi x.$$

It can, however, be proved that an approximate value of the series will be obtained by taking the aggregate of the convergent terms only.

SECTION III.

EQUATIONS OF DIFFERENCES.

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72. We now come to the case in which the relation between the principal variable and any function of it is to be determined by means of an equation between x, u_x, and one or more of the successive values, or differences, of u_x; that is, from equations of the form

$$F(x, u_x, u_{x+1}, \dots u_{x+n}) = 0,$$

$$\text{or, } f(x, u_x, \Delta u_x, \dots \Delta^n u_x) = 0,$$

since, by the theorems of Arts. 18 and 23, these forms are convertible one into the other. An Equation of Differences is said to be of the n^{th} order when the successive value, or the difference, of the highest order which it involves is the n^{th}.

73. The complete integral of an equation of differences of the n^{th} order will contain n arbitrary constants.

Let $\dots u_x, u_{x+h}, u_{x+2h}, \dots$ be a series of terms corresponding to the successive values x, $x+h$, $x+2h, \dots$; and let

$$F(x, u_x, a) = 0$$

be the equation by which the general term is determined as a function of x and a, or the equation of the series, a being an arbitrary constant. Since this equation must hold for all the succeeding terms, we shall have

$$F(x+h, u_{x+h}, a) = 0.$$

Eliminating (a) between these two equations, we get an equation between x, u_x, and u_{x+h}; or, substituting $u_x + \Delta u_x$ for u_{x+h}, an equation between x, u_x, and Δu_x, which is the equation of differences of the first order whose primitive equation is

$$F(x, u_x, a) = 0.$$

In like manner if the equation of the general term contained two arbitrary constants a and b, as

$$F(x, u_x, a, b) = 0,$$

we might eliminate a and b by means of the two succeeding equations,

$$F(x + h, u_{x+h}, a, b) = 0, \qquad F(x + 2h, u_{x+2h}, a, b) = 0,$$

and thus get an equation between x, u_x, u_{x+h}, u_{x+2h}; or, substituting

$$u_x + \Delta u_x \text{ for } u_{x+h}, \text{ and } u_x + 2\Delta u_x + \Delta^2 u_x \text{ for } u_{x+2h},$$

an equation between x, u_x, Δu_x, $\Delta^2 u_x$, without the constants a and b, which is an equation of differences of the second order, having for its complete primitive the equation

$$F(x, u_x, a, b) = 0.$$

Hence it appears that every equation of differences of the first order, or between two successive terms, will introduce one arbitrary constant into the equation of the series; every equation of differences of the second order, or between three successive terms, will introduce two arbitrary constants into the equation of the series; and, generally, every equation of differences of the n^{th} order will introduce n arbitrary constants into the equation of the series.

Linear Equation of Differences of the First Order.

74. The general equation of the first order and degree is

$$u_{x+1} - A_x u_x = B_x,$$

A_x and B_x being functions of x. To integrate it, assume

$$u_x = v_x w_x,$$

$$\therefore \quad v_{x+1}(w_x + \Delta w_x) - A_x v_x w_x = B_x;$$

and in order that this equation may resolve itself into two others, each of which admits of being integrated, assume (as we are at liberty to do, having made only one supposition respecting v_x and w_x)

$$v_{x+1} w_x - A_x v_x w_x = 0,$$

or, dividing by $v_x w_x$,

$$\frac{v_{x+1}}{v_x} = A_x.$$

But $\Delta \log v_x = \log \frac{v_{x+1}}{v_x}$, $\quad \therefore \quad \Delta \log v_x = \log A_x$,

$$\therefore \quad \log v_x = \Sigma \log A_x = \log PA_{x-1}, \quad \text{(Art. 50.)}$$

or $\quad v_x = PA_{x-1}$, the constant being unnecessary.

The other part of the equation gives

$$v_{x+1} \Delta w_x = B_x,$$

$$\therefore \quad \Delta w_x = \frac{B_x}{PA_x},$$

$$\therefore \quad w_x = \Sigma \left(\frac{B_x}{PA_x}\right) + C,$$

and $\quad u_x = PA_{x-1} \left\{\Sigma \left(\frac{B_x}{PA_x}\right) + C\right\}$

the complete integral, involving one arbitrary constant.

Obs. Taking the difference of the result

$$\frac{u_x}{PA_{x-1}} = \Sigma\left(\frac{B_x}{PA_x}\right) + C, \text{ we get } \frac{u_{x+1} - A_x u_x}{PA_x} = \frac{B_x}{PA_x};$$

which shews that $\frac{1}{PA_x}$ is a factor which makes each side of the proposed equation integrable; and it is generally the most convenient way of integrating the equation to multiply it by this factor.

Ex. 1. $u_{x+1} - au_x = x^2.$

Here $A_x = a, \quad PA_x = a^x;$

$$\therefore \frac{u_x}{a^{x-1}} = \Sigma\left(\frac{x^2}{a^x}\right) = \Sigma(x^2\alpha^x), \quad \text{putting } \frac{1}{a} = \alpha,$$

$$= \frac{x^2\alpha^x}{\alpha - 1} - \frac{(2x+1)\alpha^{x+1}}{(\alpha-1)^2} + \frac{2\alpha^{x+2}}{(\alpha-1)^3} + C; \text{ (Art. 59.)}$$

$$\therefore u_x = \frac{x^2}{1-a} - \frac{2x+1}{(1-a)^2} + \frac{2}{(1-a)^3} + Ca^{x-1}.$$

Ex. 2. $u_{x+1} - au_x = \cos x\theta.$

$$u_x = \frac{\cos(x-1)\theta - a\cos x\theta}{a^2 - 2a\cos\theta + 1} + Ca^x.$$

Ex. 3. Two vessels which hold (a) and (b) gallons respectively are filled, the one with proof spirit, the other with water; (c) gallons are taken from each and poured into the other; and this is repeated such a number of times as to make their contents of the same strength; find the number of times.

$$x = \frac{\log\frac{1}{2}\left(1 - \frac{a}{b}\right)}{\log\left(1 - \frac{c}{a} - \frac{c}{b}\right)}.$$

Indirect Integrals of Equations of Differences.

75. Since the equation of differences of the first order,

$$f(x, u_x, \Delta u_x) = 0,$$

is formed by eliminating the constant (a) between the equations

$$u_x = F(x, a), \quad u_{x+h} = F(x + h, a),$$

it follows that we shall arrive at the same equation of differences, whether a be constant, or be a function of x, as a_x, provided it be such that

$$F(x + h, a_{x+h}) = F(x + h, a_x). \quad (1.)$$

Now this equation is satisfied by $a_{x+h} = a_x$, which gives $\Delta a_x = 0$, and $a_x = a$, a constant, and leads to the ordinary or direct equation to the series,

$$u_x = F(x, a).$$

Also equation (1) will be divisible by $a_{x+h} - a_x$, because a^x is a value of a_{x+h}, which satisfies equation (1); and if dimensions of a_{x+h} and a_x superior to the first are involved in it, the result of this division will be an equation involving a_{x+h} and a_x; i. e., an equation of differences of the first order with respect to a_x, the solution of which will give one or more values of a_x in terms of x and arbitrary constants; and these being substituted for a_x in the equation

$$u_x = F(x, a_x),$$

will furnish equations of series, which are primitive equations of

$$f(x, u_x, \Delta u_x) = 0,$$

and each involves an arbitrary constant.

If equation (1) does not involve higher dimensions of a_x and a_{x+h} than the first, they will disappear from the result when it is divided by $a_{x+h} - a_x$. In this case a_x will have only one value, viz. $a_x = a$, and there will be only one equation of a series corresponding to the proposed equation of differences. The mode in which the indirect solutions just treated of are obtained, is analogous to that in which the singular solutions of differential equations are obtained; but whereas the latter can contain no arbitrary constant, indirect solutions of equations of differences may contain as many arbitrary constants as the complete integral itself from which they are deduced.

Ex. $u_x = x\Delta u_x + F(\Delta u_x)$.

Taking the difference, we find

$$\Delta u_x = \Delta u_x + (x+1)\,\Delta^2 u_x + \Delta F(\Delta u_x),$$

$$\text{or}\quad 0 = (x+1)\,\Delta^2 u_x + \Delta F(\Delta u_x),$$

which is evidently satisfied by $\Delta u_x = a$, a constant;

$$\therefore\quad u_x = ax + a';$$

and substituting in the proposed equation

$$ax + a' = ax + F(a), \qquad \therefore\quad a' = F(a);$$

$$\therefore\quad u_x = xa + F(a),$$

the complete integral, containing one arbitrary constant.

For the indirect solutions we shall have

$$u_x = xa_x + F(a_x),$$

a_x being determined from the equation

$$(x+1)\,a_{x+1} + F(a_{x+1}) = (x+1)\,a_x + F(a_x).$$

Suppose, for instance, that $F(a_x) = a_x^2$,

$$\therefore \quad (x+1)(a_{x+1} - a_x) + a_{x+1}^2 - a_x^2 = 0,$$

or, rejecting the factor $a_{x+1} - a_x$,

$$a_{x+1} + a_x = -(x+1);$$

$$\therefore \quad \frac{a_x}{(-1)^{x-1}} = -\Sigma(x+1)(-1)^x = \frac{2x+1}{4}(-1)^x + C,$$

$$\text{or} \quad a_x = -\frac{2x+1}{4} + C(-1)^x;$$

$$\therefore \quad u_x = xa_x + a_x^2 = \frac{1-4x^2}{16} - \tfrac{1}{2}C(-1)^x + C^2.$$

Linear Equations of Differences of all Orders.

76. The linear equation of Differences of the n^{th} order is

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \ldots + p_n u_x = X,$$

all the coefficients being functions of x; the first step towards its integration is to establish the following theorem.

77. If there be n particular values

$${}^1u_x, \quad {}^2u_x, \ldots {}^nu_x,$$

which, when substituted for u_x, satisfy the equation

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \ldots + p_n u_x = 0,$$

that has no term independent of u_x, its complete integral is

$$u_x = a_1\, {}^1u_x + a_2\, {}^2u_x + \ldots + a_n\, {}^nu_x,$$

$a_1, \quad a_2, \ldots a_n$ being arbitrary constants.

For let these values be substituted in the expression

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \dots + p_n u_x,$$

and it becomes, (collecting the terms multiplied by the factors $a_1, a_2, \dots a_n$)

$$a_1 ({}^1u_{x+n} + p_1 {}^1u_{x+n-1} + \dots + p_n {}^1u_x) + a_2 ({}^2u_{x+n} + p_1 {}^2u_{x+n-1} + \dots$$
$$+ p_n {}^2u_x) + \dots + a_n ({}^nu_{x+n} + p_1 {}^nu_{x+n-1} + \dots + p_n {}^nu_x).$$

Now since ${}^1u_x, {}^2u_x, \dots {}^nu_x$, satisfy the proposed equation, each of the quantities included within brackets is equal to zero, therefore the whole is identically zero; consequently the assumed value of u_x satisfies the proposed equation, and it contains n arbitrary constants, therefore it is the complete integral of that equation.

78. To integrate the equation of differences,

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \dots + p_n u_x = q,$$

all the coefficients and q being constants.

Assume $u_x = v_x + k$; then by substitution we get

$$v_{x+n} + p_1 v_{x+n-1} + p_2 v_{x+n-2} + \dots$$
$$+ p_n v_x + k (1 + p_1 + p_2 + \dots + p_n) - q = 0.$$

Let $k = \dfrac{q}{1 + p_1 + p_2 + \dots + p_n}$, then the equation becomes

$$v_{x+n} + p_1 v_{x+n-1} + p_2 v_{x+n-2} + \dots + p_n v_x = 0. \quad (1.)$$

Let $v_x = a^x$, then $a^x (a^n + p_1 a^{n-1} + p_2 a^{n-2} + \dots + p_n)$ is the value of the first member; now this will vanish if a be any root of the equation (called the auxiliary equation),

$$f(a) = a^n + p_1 a^{n-1} + p_2 a^{n-2} + \dots + p_{n-1} a + p_n = 0.$$

Hence the (n) roots of this equation $a_1, a_2, a_3, \ldots a_n$ will give n particular values of v_x, a_1^x, a_2^x, $a_3^x \ldots a_n^x$ which satisfy equation (1); therefore its complete integral is

$$v_x = c_1 a_1^x + c_2 a_2^x + \ldots + c_n a_n^x;$$

and the complete integral of the proposed equation is

$$u_x = c_1 a_1^x + c_2 a_2^x + \ldots + c_n a_n^x + \frac{q}{1 + p_1 + p_2 + \ldots + p_n}.$$

79. If the auxiliary equation have equal roots, the above ceases to be the form of the complete solution; because, in that case, it does not involve the due number of arbitrary constants; and it must be modified as follows. Suppose two roots a_1, a_2, to be very nearly equal to one another, so that $a_2 = a_1 + h$, h being a very small known quantity;

then,

$$c_1 a_1^x + c_2 a_2^x = (c_1 + c_2)\, a_1^x + c_2 \left\{x a_1^{x-1} h + \frac{x(x-1)}{1.2} a_1^{x-2} h^2 + \&c.\right\}$$

$$= C_1 a_1^x + C_2 \left\{x a_1^x + \frac{x(x-1)}{1.2} a_1^{x-1} h + \&c.\right\}$$

replacing the constants

$$c_1 + c_2 \text{ by } C_1, \text{ and } \frac{h c_2}{a_1} \text{ by } C_2;$$

now this continues true however small h be taken, and therefore when $h = 0$, when it becomes

$$(C_1 + C_2 x)\, a_1^x;$$

$$\therefore\ u_x = (C_1 + C_2 x)\, a_1^x + c_3 a_3^x + \&c.$$

Similarly, if the auxiliary equation have r roots equal to a_1, the complete solution will be

$$u_x = (c_0 + c_1 x + c_2 x^2 + \ldots + c_{r-1} x^{r-1})\, a_1^x + c_{r+1} a_{r+1}^x + \ldots + c_n a_n^x;$$

of the correctness of which, we may be assured by the following reverse process. Assume $u_x = a^x v_x$, then

$$u_{x+n} = a^x \,.\, a^n (1+\Delta)^n v_x,$$

and the first side of the equation becomes, when divided by a^x,

$$a^n (1+\Delta)^n v_x + p_1 a^{n-1} (1+\Delta)^{n-1} v_x + \&c. + p_n v_x$$

$$= \{(a + a\Delta)^n + p_1 (a + a\Delta)^{n-1} + \&c. + p_n\} v_x$$

$$= f(a + a\Delta) v_x = \{f(a) + f'(a) \,.\, a\Delta + f''(a) \frac{a^2\Delta^2}{1\,.\,2} + \&c.\} v_x$$

$$= f(a) \,.\, v_x + \frac{a}{1} \,.\, f'(a) \,.\, \Delta v_x + \frac{a^2}{1\,.\,2} f''(a) \,.\, \Delta^2 v_x + \&c. + a^n \Delta^n v_x.$$

Now suppose $a = a_1$, and $f(a) = 0$ to have r roots equal to a_1; this makes the terms as far as $f^{(r-1)}(a)$ vanish; and if

$$v_x = c_0 + c_1 x + \ldots + c_{r-1} x^{r-1},$$

$$\text{then} \quad \Delta^r v_x = 0, \quad \Delta^{r+1} v_x = 0, \quad \&c.,$$

and all the remaining terms vanish; and consequently the equation is satisfied by

$$u_x = (c_0 + c_1 x + \ldots + c_{r-1} x^{r-1})\, a_1^x.$$

80. Also if the auxiliary equation have a pair of imaginary roots,

$$m \pm n\sqrt{-1} = \rho (\cos\theta \pm \sqrt{-1} \sin\theta),$$

$$\text{putting} \quad \rho = \sqrt{m^2 + n^2}, \quad \tan\theta = \frac{n}{m},$$

the corresponding terms in the value of u_x will be

$$C\rho^x (\cos\theta + \sqrt{-1} \sin\theta)^x + C'\rho^x (\cos\theta - \sqrt{-1} \sin\theta)^x$$

$$= \rho^x (c_1 \cos x\theta + c_2 \sin x\theta),$$

changing the arbitrary constants.

And if there be r pairs of imaginary roots, the corresponding terms in the value of u_x will be

$$(a_0 + a_1 x + a_2 x^2 + \ldots + a_{r-1} x^{r-1})\, \rho^x (\cos\theta + \sqrt{-1}\sin\theta)^x$$
$$+ (b_0 + b_1 x + \ldots + b_{r-1} x^{r-1})\, \rho^x (\cos\theta - \sqrt{-1}\sin\theta)^x;$$

or, changing the arbitrary constants,

$$(c_0 + c_1 x + \ldots + c_{r-1} x^{r-1})\, \rho^x \cos x\theta$$
$$+ (c'_0 + c'_1 x + \ldots + c'_{r-1} x^{r-1})\, \rho^x \sin x\theta.$$

Obs. The Differential Calculus being a particular case of that of Finite Differences, a strict analogy exists between the methods and results in the two subjects, as the reader cannot fail to have observed; indeed whenever in the latter a result is obtained with an indeterminate increment for the principal variable, it is possible to pass to the corresponding result in the former, by a method similar to that pursued in Art. 25; and which we shall further illustrate by the following instance.

In the equation $\quad u_{x+2h} - 2m u_{x+h} + (m^2 + n^2)\, u_x = 0,$

putting $u_x = a^x$, we find $a = (m \pm n\sqrt{-1})^{\frac{1}{h}}$;

$$\therefore \quad u_x = C_1 (m + n\sqrt{-1})^{\frac{x}{h}} + C_2 (m - n\sqrt{-1})^{\frac{x}{h}}$$

$$= (m^2 + n^2)^{\frac{x}{2h}} \left\{ c_1 \cos\left(\frac{x}{h}\tan^{-1}\frac{n}{m}\right) + c_2 \sin\left(\frac{x}{h}\tan^{-1}\frac{n}{m}\right) \right\}.$$

Now the proposed equation is

$$\Delta^2 u_x - 2(m-1)\,\Delta u_x + \left\{(m-1)^2 + n^2\right\} u_x = 0,$$

or if we replace the known quantities $m - 1$ and n by other known quantities $m_1 h$ and $n_1 h$, the equation and its solution take the forms

$$\frac{\Delta^2 u_x}{h^2} - 2m_1 \frac{\Delta u_x}{h} + (m_1^2 + n_1^2)\, u_x = 0,$$

$$u_x = \{(1+m_1h)^2 + n_1^2h^2\}^{\frac{x}{2h}} \left\{c_1 \cos\left(\frac{x}{h}\tan^{-1}\frac{n_1h}{1+m_1h}\right)\right.$$
$$\left. + c_2 \sin\left(\frac{x}{h}\tan^{-1}\frac{n_1h}{1+m_1h}\right)\right\}.$$

Now take the limits of these expressions when $h=0$, and we find the well known results, since $(1+2m_1h)^{\frac{1}{2h}}$ becomes e^{m_1},

$$d_x^2u_x - 2m_1d_xu_x + (m_1^2+n_1^2)u_x = 0,$$

$$u_x = e^{m_1x}(c_1\cos n_1x + c_2\sin n_1x).$$

81. To integrate the linear equation of differences of the n^{th} order,

$$u_{x+n} + p_1u_{x+n-1} + p_2u_{x+n-2} + \dots + p_nu_x = X,$$

the coefficients being functions of x.

If ${}^1u_x$, ${}^2u_x$, ... ${}^nu_x$ be n particular values of v_x in the equation

$$v_{x+n} + p_1v_{x+n-1} + p_2v_{x+n-2} + \dots + p_nv_x = 0, \quad (1.)$$

with which the proposed coincides when its second member is zero, we have (Art. 77.)

$$v_x = c_1{}^1u_x + c_2{}^2u_x + \dots + c_n{}^nu_x.$$

If we now divide both sides by ${}^1u_x$, and take the difference, we shall eliminate c_1; next dividing both sides by the coefficient of c_2, which suppose ${}^1v_x$, and taking the difference, we shall eliminate c_2; again dividing by ${}^2v_x$, the coefficient of c_3, and taking the difference, we shall eliminate c_3; and proceeding in this manner till all the constants are eliminated, our final result will be of the form (each Δ affecting the whole of the expression that follows it)

$$\Delta\frac{1}{{}^{n-1}v_x}\Delta\frac{1}{{}^{n-2}v_x}\dots\Delta\frac{1}{{}^1v_x}\Delta\frac{v_x}{{}^1u_x},$$

in which expression the coefficient of v_{x+n} is evidently

$$\frac{1}{{}^{n-1}v_{x+1} \cdot {}^{n-2}v_{x+2} \ldots {}^{1}v_{x+n-1}\, {}^{1}u_{x+n}};$$

therefore, dividing by this coefficient, we get the expression

$${}^{n-1}v_{x+1}\, {}^{n-2}v_{x+2} \ldots {}^{1}v_{x+n-1}\, {}^{1}u_{x+n} \Delta \frac{1}{{}^{n-1}v_x} \Delta \frac{1}{{}^{n-2}v_x} \ldots \Delta \frac{1}{{}^{1}v_x} \Delta \frac{v_x}{{}^{1}u_x},$$

which must be equivalent to the first member of equation (1). Therefore the same expression, only with u_x instead of v_x, must be equivalent to the first member of the proposed equation, and consequently equal to X; hence, equating these equals, and integrating, we get

$$u_x = {}^{1}u_x \Sigma\, {}^{1}v_x \Sigma\, {}^{2}v_x \ldots \Sigma\, {}^{n-1}v_x\, \Sigma \frac{X}{{}^{n-1}v_{x+1}\, {}^{n-2}v_{x+2} \ldots {}^{1}v_{x+n-1}\, {}^{1}u_{x+n}},$$

(each Σ affecting the whole of the expression which follows it) which is a general formula for the integration of any linear equation of differences whatever.

82. As a first exemplification of this method, suppose the coefficients of the equation to be constant, and $a_1, a_2, \ldots a_n$ to be the n roots of its auxiliary equation.

Then $\quad v_x = c_1 a_1^x + c_2 a_2^x + c_3 a_3^x + \ldots + c_n a_n^x,$

$$\therefore \quad \Delta \frac{v_x}{a_1^x} = c_2 \left(\frac{a_2}{a_1}\right)^x + c_3 \left(\frac{a_3}{a_1}\right)^x + \ldots + c_n \left(\frac{a_n}{a_1}\right)^x,$$

changing, both in this, and in the similar succeeding steps, the arbitrary constants;

$$\Delta \left(\frac{a_1}{a_2}\right)^x \Delta \frac{v_x}{a_1^x} = c_3 \left(\frac{a_3}{a_2}\right)^x + \ldots + c_n \left(\frac{a_n}{a_2}\right)^x$$

$$\Delta \left(\frac{a_2}{a_3}\right)^x \Delta \left(\frac{a_1}{a_2}\right)^x \Delta \frac{v_x}{a_1^x} = c_4 \left(\frac{a_4}{a_3}\right)^x + \ldots + c_n \left(\frac{a_n}{a_3}\right)^x$$

....

$$\Delta \left(\frac{a_{n-1}}{a_n}\right)^x \Delta \left(\frac{a_{n-2}}{a_{n-1}}\right)^x \ldots \Delta \left(\frac{a_1}{a_2}\right)^x \Delta \frac{v_x}{a_1^x} = 0,$$

in which expression the coefficient of v_{x+n} is evidently

$$\frac{1}{a_n^x} \frac{1}{a_n a_{n-1} \dots a_2 a_1} = \frac{1}{(-1)^n p_n a_n^x},$$

therefore, dividing by this coefficient, and replacing v_x by u_x, we get

$$(-1)^n p_n a_n^x \Delta \left(\frac{a_{n-1}}{a_n}\right)^x \Delta \left(\frac{a_{n-2}}{a_{n-1}}\right)^x \dots \Delta \left(\frac{a_1}{a_2}\right)^x \Delta \frac{u_x}{a_1^x} = X,$$

$$\therefore \quad u_x = \frac{(-1)^n}{p_n} a_1^x \Sigma \left(\frac{a_2}{a_1}\right)^x \dots \Sigma \left(\frac{a_n}{a_{n-1}}\right)^x \Sigma \left(\frac{X}{a_n^x}\right);$$

or, if we choose to introduce the arbitrary constant after each integration,

$$u_x = c_1 a_1^x + c_2 a_2^x + \dots$$

$$+ c_n a_n^x + (-1)^n \frac{1}{p_n} a_1^x \Sigma \left(\frac{a_2}{a_1}\right)^x \Sigma \left(\frac{a_3}{a_2}\right)^x \dots \Sigma \left(\frac{a_n}{a_{n-1}}\right)^x \Sigma \left(\frac{X}{a_n^x}\right).^*$$

83. If the auxiliary equation contain equal, or imaginary roots, this method is still applicable; it is only necessary to assume for v_x the value belonging to the case of equal or imaginary roots, as will be seen in the following instances.

Ex. 1. To solve the equation

$$a^2 u_{x+2} - 2a u_{x+1} + u_x = X.$$

$$\text{Here } v_x = (c + c^1 x) \frac{1}{a^x};$$

$$\therefore \ \Delta^2 (a^x v_x) = 0,$$

* For this method, and the corresponding one of solving the general linear differential equation, I am indebted to Mr Gaskin.

in which expression, the coefficient of v_{x+2} is a^{x+2}; therefore, dividing by this quantity, and replacing v_x by u_x, we get

$$\frac{1}{a^{x+2}}\Delta^2(a^x u_x) = X;$$

$$\therefore\ u_x = \frac{1}{a^x}\Sigma^2(Xa^{x+2}).$$

Let $X = x$, then $\Sigma^2(xa^x) = x\Sigma^2 a^x - 2\Sigma^3 a^{x+1} + c'x + c$, (Art. 58.)

$$\therefore\ u_x = \frac{1}{a^{x-2}}\left\{\frac{xa^x}{(a-1)^2} - \frac{2a^{x+1}}{(a-1)^3} + c'x + c\right\}$$

$$= (C + C^1 x)\frac{1}{a^x} + \frac{a^2 x}{(a-1)^2} - \frac{2a^3}{(a-1)^3}.$$

Ex. 2. To solve the equation

$$u_{x+2} - 2mu_{x+1} + (m^2 + n^2)u_x = X.$$

Here $v_x = c_1\rho^x \sin x\theta + c_2\rho^x \cos x\theta$,

$$\text{where } \rho^2 = m^2 + n^2, \quad \tan\theta = \frac{n}{m};$$

$$\therefore\ \frac{v_x}{\rho^x \cos x\theta} = c_1 \tan x\theta + c_2;$$

$$\therefore\ \Delta\left(\frac{v_x}{\rho^x \cos x\theta}\right) = \frac{c_1 \sin\theta}{\cos x\theta \cos(x+1)\theta};$$

$$\therefore\ \Delta \cos x\theta \cos(x+1)\theta\,\Delta\left(\frac{v_x}{\rho^x \cos x\theta}\right) = 0,$$

in which expression, the coefficient of v_{x+2} is $\dfrac{\cos(x+1)\theta}{\rho^{x+2}}$;

therefore, dividing by this coefficient, and replacing v_x by u_x, we get

$$\frac{\rho^{x+2}}{\cos (x+1)\theta} \Delta \cos x\theta \cos (x+1)\theta \Delta \left(\frac{u_x}{\rho^x \cos x\theta}\right) = X;$$

$$\therefore\ u_x = \rho^x \cos x\theta \Sigma \sec x\theta \sec (x+1)\theta \Sigma \left\{\frac{X \cos (x+1)\theta}{\rho^{x+2}}\right\};$$

to which we may add the terms $c_1\rho^x \sin x\theta + c_2\rho^x \cos x\theta$, if we suppose a constant to be added after each integration.

Ex. 3. To solve the equation

$$u_{x+4} + p_1 u_{x+3} + p_2 u_{x+2} + p_3 u_{x+1} + p_4 u_x = X,$$

where the auxiliary equation has two pairs of imaginary roots, so that

$$a^4 + p_1 a^3 + p_2 a^2 + p_3 a + p_4 = \left\{(a-m)^2 + n^2\right\}^2.$$

Here $\quad v_x = (a + a' x)\rho^x \cos x\theta + (b + b' x)\rho^x \sin x\theta,$

$$\therefore\quad \frac{v_x}{\rho^x \cos x\theta} = a + a' x + (b + b' x) \tan x\theta,$$

$$\therefore\quad \Delta \frac{v_x}{\rho^x \cos x\theta} = a' + b' \tan x\theta + (b + b' x)\frac{\sin\theta}{\cos x\theta \cos (x+1)\theta},$$

$$\therefore\quad \cos x\theta \cos (x+1)\theta \Delta \frac{v_x}{\rho^x \cos x\theta} = \tfrac{1}{2} a' \left\{\cos (2x+1)\theta + \cos\theta\right\}$$

$$+ \tfrac{1}{2} b' \left\{\sin (2x+1)\theta - \sin\theta\right\} + (b + b' x)\sin\theta,$$

$$\therefore\quad \Delta \cos x\theta \cos (x+1)\theta \Delta \frac{v_x}{\rho^x \cos x\theta} = -a' \sin 2(x+1)\theta \sin\theta$$

$$+ b' \cos 2(x+1)\theta \sin\theta + b' \sin\theta,$$

$$\therefore\quad \sec^2 (x+1)\theta \Delta \cos x\theta \cos (x+1)\theta \Delta \frac{v_x}{\rho^x \cos x\theta}$$

$$= -2a' \tan (x+1)\theta \sin\theta + 2b' \sin\theta,$$

$$\therefore \ \Delta \sec^2 (x+1)\theta \, \Delta (\qquad) = -\frac{2 a' \sin^2 \theta}{\cos (x+1)\theta \cos (x+2)\theta},$$

$$\therefore \ \Delta \cos (x+1)\theta \cos (x+2)\theta \, \Delta \sec^2 (x+1)\theta \times$$

$$\Delta \cos x\theta \cos (x+1)\theta \, \Delta \frac{v_x}{\rho^x \cos x\theta} = 0,$$

in which expression, the coefficient of v_{x+4} is $\dfrac{\cos (x+2)\theta}{\rho^{x+4}}$; therefore, dividing by this coefficient, and replacing v_x by u_x, we get an expression equivalent to the first member of the proposed equation; hence, equating it to X, and integrating, we find

$$u_x = \rho^x \cos x\theta \, \Sigma \sec x\theta \sec (x+1)\theta \, \Sigma \cos^2 (x+1)\theta \times$$

$$\Sigma \sec (x+1)\theta \sec (x+2)\theta \, \Sigma \left\{ \frac{X \cos (x+2)\theta}{\rho^{x+4}} \right\}.$$

84. Having given a particular integral of the equation

$$u_{x+2} + A_x u_{x+1} + B_x u_x = 0,$$

to find its complete solution.

Let v_x be a particular integral, then

$$v_{x+2} + A_x v_{x+1} + B_x v_x = 0;$$

Hence, eliminating A_x, we find, putting $\dfrac{u_x}{v_x} = z_x$,

$$\frac{\Delta z_{x+1}}{\Delta z_x} = \frac{B_x v_x}{v_{x+2}} = w_x, \text{ suppose;}$$

$$\text{or} \quad \Delta \log (\Delta z_x) = \log w_x,$$

$$\therefore \ \Delta z_x = C P w_{x-1}, \quad \text{(Art. 50.)}$$

$$\therefore \ u_x = C v_x \Sigma (P w_{x-1}).$$

85. If we know a particular integral of a linear equation of any order that has no term independent of u_x, we may reduce it to another equation of the same kind of the order immediately inferior.

Let $u_x = v_x$ be a particular integral of a linear equation of differences of the n^{th} order reduced to the form

$$\Delta^n u_x + q_1 \Delta^{n-1} u_x + q_2 \Delta^{n-2} u_x + \dots + q_n u_x = f(\Delta)\, u_x = 0 \quad (1).$$

Assume $\quad u_x = v_x \Sigma w_x$;

$$\text{then } \Delta^n u_x = \{\Delta + (1+\Delta)\,\Delta'\}^n\, v_x \Sigma w_x,$$

where Δ affects v_x only, and Δ' affects Σw_x only, (Art. 34.), and the proposed equation becomes

$$f\{\Delta + (1+\Delta)\,\Delta'\}\, v_x \Sigma w_x = 0.$$

Now $f\{\Delta + (1+\Delta)\,\Delta'\}$ is a rational integral function of Δ and Δ', which we wish to arrange according to powers of Δ'; and this may be done at once by Taylor's theorem, which gives

$$f(\Delta) + \Delta' f_1(\Delta)(1+\Delta) + \frac{\Delta'^2}{1.2} f_2(\Delta)(1+\Delta)^2 + \dots + \Delta'^n (1+\Delta)^n.$$

Hence, observing that $f(\Delta)\, v_x = 0$, since v_x substituted for u_x satisfies the equation, and that $(1+\Delta)^r v_x = v_{x+r}$, we get the depressed equation

$$w_x f_1(\Delta) v_{x+1} + \frac{1}{1.2} \Delta w_x f_2(\Delta)\, v_{x+2} + \dots + \Delta^{n-1} w_x \,.\, v_{x+n} = 0;$$

or, reversing the order of the terms, (since $f_r(\Delta)$ means the same function of Δ, that $d_x^r f(x)$ does of x),

$$v_{x+n} \Delta^{n-1} w_x + (n \Delta v_{x+n-1} + q_1 v_{x+n-1})\, \Delta^{n-2} w_x$$

$$+ \left\{\frac{n(n-1)}{1.2} \Delta^2 v_{x+n-2} + (n-1) q_1 \Delta v_{x+n-2} + q_2 v_{x+n-2}\right\} \Delta^{n-3} w_x + \dots$$

$$+ f_1(\Delta)\, v_{x+1} \,.\, w_x = 0, \quad (2.)$$

a linear equation of the $(n-1)^{\text{th}}$ order, of the same form as the original one. Similarly, if we know another particular value of u_x, z_x, then $\Delta\left(\frac{z_x}{v_x}\right)$ will be a value of w_x in equation (2), which may be depressed to another of the same form of the $(n-2)^{\text{th}}$ order; and if we know r particular solutions of equation (1), we may in this way depress it to an equation of the same form of the $(n-r)^{\text{th}}$ order. As a linear equation of differences of the first order and degree can always be solved, it appears that to obtain the complete integral of a linear equation of the n^{th} order, we must know $n-1$ particular solutions.

86. Besides linear equations of differences, and such equations as can be reduced to that form, very little is known of equations of differences of the second and higher orders. The following are instances of equations which admit of reduction to linear equations with constant coefficients.

1. $$u_{x+n} + p_1 v_{x+n} u_{x+n-1} + p_2 v_{x+n} v_{x+n-1} u_{x+n-2} + \dots$$
$$+ p_n v_{x+n} v_{x+n-1} \dots v_{x+1} u_x = 0;$$

where p_1, p_2, &c. are constants, and v_x is any function of x. Assume $u_x = w_x P v_x$, then the equation becomes divisible by $P v_{x+n}$, and is reduced to the linear equation with constant coefficients,

$$w_{x+n} + p_1 w_{x+n-1} + p_2 w_{x+n-2} + \dots + p_n w_x = 0.$$

2. $$u_{x+n} + p_1 a^x u_{x+n-1} + p_2 a^{2x} u_{x+n-2} + \dots + p_n a^{nx} u_x = 0.$$

Assume $u_x = v_x a^{(1+2+3\dots+x)}$, then any term $p_r a^{rx} u_{x+n-r}$ becomes

$$= p_r v_{x+n-r} a^{\frac{1}{2}x(x+2n+1)} \cdot a^{\frac{1}{2}(n-r)(n-r+1)};$$

therefore the equation becomes divisible by $a^{\frac{1}{2}x(x+2n+1)}$, and is reduced to the linear equation with constant coefficients,

$$a^{\frac{1}{2}(n+1)n} v_{x+n} + p_1 a^{\frac{1}{2}n(n-1)} v_{x+n-1} + p_2 a^{\frac{1}{2}(n-1)(n-2)} v_{x+n-2} + \dots + p_n v_x = 0.$$

3. $u_{x+2} + \{a + b(-1)^x\} u_{x+1} + c u_x = 0.$

$$\text{Let} \quad u_x \;= v_x \sqrt{a + b(-1)^x},$$

$$\text{then} \quad u_{x+1} = v_{x+1} \sqrt{a - b(-1)^x},$$

$$u_{x+2} = v_{x+2} \sqrt{a + b(-1)^x};$$

$$\therefore \quad v_{x+2} + \sqrt{a^2 - b^2}\, v_{x+1} + c v_x = 0.$$

4. $u_{x+1} u_x + a u_{x+1} + b u_x + c = 0.$

Assume $u_x + a = \dfrac{v_{x+1}}{v_x}$, then $u_{x+1} + a = \dfrac{v_{x+2}}{v_{x+1}}$,

$$\therefore \quad \frac{v_{x+1}}{v_x} \left(\frac{v_{x+2}}{v_{x+1}} - a \right) + b \left(\frac{v_{x+1}}{v_x} - a \right) + c = 0,$$

$$\text{or} \quad v_{x+2} - (a - b)\, v_{x+1} + (c - ab)\, v_x = 0.$$

The two arbitrary constants which will appear in the value of v_x, must be reduced to a single constant, by the condition of the proposed equation being satisfied.

5. $u_{x+1} - 2 u_x^2 + 1 = 0.$

Assume $u_x = \cos v_x$, then $u_x = \frac{1}{2} (c^{2^x} + c^{-2^x})$.

Prob. 1. To find the value of

$$\sqrt{2 - \sqrt{2 - \sqrt{2 - \dots}}} \text{ to } x \text{ terms},$$

$$\text{let it} = u_x, \quad \text{then} \quad u_{x+1} = \sqrt{2 - u_x};$$

$$\therefore \quad u_{x+1}^2 + u_x = 2; \quad \text{let} \quad u_x = 2 \cos v_x,$$

$$\text{then} \quad 4 \cos^2 v_{x+1} + 2 \cos v_x = 2;$$

$$\therefore \quad 2 \cos^2 v_{x+1} = 1 - \cos v_x = 2 \sin^2 \tfrac{1}{2} v_x;$$

$$\therefore \quad \cos v_{x+1} = \sin \tfrac{1}{2} v_x = \cos \left(\frac{\pi}{2} - \tfrac{1}{2} v_x \right);$$

$$\therefore \quad v_{x+1} + \tfrac{1}{2} v_x = \frac{\pi}{2};$$

$$\therefore \quad v_x = c\,(-\tfrac{1}{2})^x + \frac{\pi}{3};$$

$$\therefore \quad u_x = 2\cos\left\{c\,(-\tfrac{1}{2})^x + \frac{\pi}{3}\right\};$$

$$\therefore \quad u_1 = 2\cos\left(-\frac{c}{2} + \frac{\pi}{3}\right) = \sqrt{2} = 2\cos\frac{\pi}{4}, \quad \therefore\ c = \frac{\pi}{6};$$

$$\therefore \quad u_x = 2\cos\left\{\frac{\pi}{6}(-\tfrac{1}{2})^x + \frac{\pi}{3}\right\} = 2\sin\frac{\pi}{6}\left\{1 - (-\tfrac{1}{2})^x\right\}.$$

PROB. 2. A swan breeds three cygnets in its second year, and four every succeeding year; and the young ones all breed according to the same law; required the number at the end of the x^{th} year.

Let u_x = number at end of x^{th} year;

then in $(x+1)^{\text{th}}$ year, there will be bred three each by those in their second year, and four each by all the others;

$$\therefore \quad u_{x+1} = u_x + 3(u_{x-1} - u_{x-2}) + 4u_{x-2} = u_x + 3u_{x-1} + u_{x-2};$$

$$\therefore \quad u_{x+3} - u_{x+2} - 3u_{x+1} - u_x = 0;$$

$$\therefore \quad m^3 - m^2 - 3m - 1 = m(m^2 - 1) - (m+1)^2 = 0;$$

$$\therefore \quad m = -1, \quad \text{and} \quad 1 \pm \sqrt{2};$$

$$\therefore \quad u_x = a(-1)^x + b(1+\sqrt{2})^x + c(1-\sqrt{2})^x,$$

$$u_0 = 0 = a + b + c,$$

$$u_1 = 1 = -a + b(1+\sqrt{2}) + c(1-\sqrt{2}),$$

$$u_2 = 4 = a + b(3 + 2\sqrt{2}) + c(3 - 2\sqrt{2});$$

$$\therefore \quad a = 1, \quad b = \tfrac{1}{2}\left(\frac{3}{\sqrt{2}} - 1\right), \qquad c = -\tfrac{1}{2}\left(\frac{3}{\sqrt{2}} + 1\right);$$

$$\therefore \quad u_x = (-1)^x + \tfrac{1}{2}\left(\frac{3}{\sqrt{2}} - 1\right)(1+\sqrt{2})^x - \tfrac{1}{2}\left(\frac{3}{\sqrt{2}} + 1\right)(1-\sqrt{2})^x.$$

If the law of increase be only one the second, and every succeeding year,

$$\sqrt{5}\, u_x = \left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1-\sqrt{5}}{2}\right)^x.$$

87. The following problems relative to continued fractions, give rise to equations of differences, which can be integrated as linear equations.

1. To determine the value of the continued fraction

$$u_x = \frac{c}{a+}\,\frac{c}{a+\dots}\,\frac{c}{a} \text{ to } x \text{ fractional terms.}$$

$$\therefore \quad u_{x+1} = \frac{c}{a + u_x}, \quad \text{or} \quad u_{x+1}(u_x + a) = c;$$

$$\text{assume} \quad u_x + a = \frac{v_{x+1}}{v_x}, \text{ then } \frac{v_{x+1}}{v_x}\left(\frac{v_{x+2}}{v_{x+1}} - a\right) = c,$$

$$\text{or} \quad v_{x+2} - a\,v_{x+1} - c\,v_x = 0;$$

$$\therefore \; v_x = c_1\alpha^x + c_2\beta^x, \; \alpha \text{ and } \beta \text{ being roots of } k^2 - ak - c = 0;$$

$$\therefore \; u_x = \frac{c_1\alpha^{x+1} + c_2\beta^{x+1}}{c_1\alpha^x + c_2\beta^x} - a = \frac{c_1\alpha^x(\alpha - a) + c_2\beta^x(\beta - a)}{c_1\alpha^x + c_2\beta^x}$$

$$= -\alpha\beta\,\frac{c_1\alpha^{x-1} + c_2\beta^{x-1}}{c_1\alpha^x + c_2\beta^x}, \text{ since } \alpha + \beta = a;$$

$$= c\,.\,\frac{\alpha^{x-1} + c_3\beta^{x-1}}{\alpha^x + c_3\beta^x}, \text{ putting } c_3 = \frac{c_2}{c_1}.$$

$$\therefore\ u_1 = \frac{c}{a} = c\,\frac{1 + c_3}{a + c_3\beta}, \quad \therefore\ c_3 = -\frac{\beta}{a},$$

$$\therefore\ u_x = c\,\frac{a^x - \beta^x}{a^{x+1} - \beta^{x+1}}.$$

If a and β be imaginary, so that

$$k^2 - ak - c = k^2 - 2k\rho\cos\theta + \rho^2,$$

$$\text{then}\quad u_x = -\rho\,\frac{\sin x\theta}{\sin(x+1)\theta}.$$

2. To find the value of the continued fraction

$$u_x = \frac{c}{a+}\ \frac{c}{a+\ldots}\ \frac{c}{a+}\ \frac{c}{b},$$

to x fractional terms, the last term being $\dfrac{c}{b}$.

Proceeding as in the last example, we find

$$u_x = c\,.\,\frac{a^{x-1} + c_3\beta^{x-1}}{a^x + c_3\beta^x},$$

$$\text{and}\quad u_1 = \frac{c}{b} = c\,.\,\frac{1 + c_3}{a + c_3\beta}, \quad \therefore\ c_3 = -\frac{b-a}{b-\beta},$$

$$\therefore\quad u_x = c\,.\,\frac{b(a^{x-1} - \beta^{x-1}) + c(a^{x-2} - \beta^{x-2})}{b(a^x - \beta^x) + c(a^{x-1} - \beta^{x-1})}.$$

3. To find the value of the continued fraction to x fractional terms,

$$u_x = \frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{a+}\ \frac{1}{b+\ldots}$$

$$\text{Here}\quad u_{x+2} = \cfrac{1}{a + \cfrac{1}{b + u_x}}, \quad \therefore\ \frac{1}{u_{x+2}} - a = \frac{1}{b + u_x},$$

$$\text{or}\quad \frac{u_{x+2}}{1 - a u_{x+2}} = b + u_x.$$

Assume $1 - a\,u_x = \dfrac{v_{x-1}}{v_{x+1}}$, $\therefore\ 1 - a\,u_{x+2} = \dfrac{v_{x+1}}{v_{x+3}}$,

$$\therefore\ \frac{1}{a}\left(1 - \frac{v_{x+1}}{v_{x+3}}\right)\frac{v_{x+3}}{v_{x+1}} = b + \frac{1}{a}\left(1 - \frac{v_{x-1}}{v_{x+1}}\right),$$

$$\therefore\ v_{x+3} - v_{x+1} = a\,b\,v_{x+1} + v_{x+1} - v_{x-1},$$

$$\text{or}\quad v_{x+3} - (2 + a\,b)\,v_{x+1} + v_{x-1} = 0,$$

of which the auxiliary equation is

$$k^4 - (2 + a\,b)\,k^2 + 1 = (k^2 - \sqrt{a\,b}\,k - 1)(k^2 + \sqrt{a\,b}\,k - 1) = 0.$$

Let $a,\ -\dfrac{1}{a},\ -a,\ \dfrac{1}{a}$, be the roots of this equation, then

$$v_x = a^x\left\{c_1 + c_2(-1)^x\right\} + \frac{1}{a^x}\left\{c_3 + c_4(-1)^x\right\},$$

$$\therefore\ a\,u_x = 1 - \frac{a^{x-1} + p_x\,a^{-(x-1)}}{a^{x+1} + p_x\,a^{-(x+1)}},$$

where $p_x = \dfrac{c_3 + c_4(-1)^{x-1}}{c_1 + c_2(-1)^{x-1}} = C + C'(-1)^{x-1}$;

$$\therefore\ a\,u_1 = 1 = 1 - \frac{1 + p_1}{a^2 + p_1 a^{-2}},\quad \therefore\ p_1 = -1 = C + C',$$

$$a\,u_2 = \frac{a\,b}{1 + a\,b} = 1 - \frac{a + p_2\,a^{-1}}{a^3 + p_2\,a^{-3}},$$

$$\therefore\ a\,b + 1 = a^2 - 1 + a^{-2} = \frac{a^3 + p_2\,a^{-3}}{a + p_2\,a^{-1}},\quad \because\ a - \frac{1}{a} = \sqrt{a\,b};$$

$$\therefore\ p_2 = 1 = C - C',\quad \therefore\ C = 0,\ C' = -1,\ \text{and}\ p_x = (-1)^x,$$

$$\therefore\ a\,u_x = 1 - \frac{a^{x-1} + (-1)^x\,a^{-x+1}}{a^{x+1} + (-1)^x\,a^{-x-1}}.$$

4. Suppose b to be essentially negative and $= -c$, so that the continued fraction is

$$\frac{1}{a+}\ \frac{1}{-c+}\ \frac{1}{a+}\ \frac{1}{-c+\ldots},$$

and let $\sqrt{ab} = \sqrt{-1}\ \sqrt{ac} = 2\sqrt{-1}\ \sin\theta$,

$$\text{then}\ \ k^2 - 2\sqrt{-1}\ \sin\theta k - 1 = 0,$$

$$\therefore\ a = \cos\theta + \sqrt{-1}\ \sin\theta;$$

$$\therefore\ au_x = 1 - \frac{\sin(x-1)\theta}{\sin(x+1)\theta},\quad \text{or}\ 1 - \frac{\cos(x-1)\theta}{\cos(x+1)\theta},$$

according as x is odd, or even.

SECTION IV.

SUMMATION OF SERIES.

Integration of the General Term.

88. One of the most direct and important applications of the Calculus of Finite Differences, is the general method which it furnishes of assigning the sum of any number of Terms of a Series of the general term of which we are able to take the integral. There will be two cases to consider, according as the general term is given explicitly in terms of the index, or is only given by means of an equation of differences. We shall begin with the former, by shewing that the sum of any number of terms ending with the general term u_x, is equal to the integral of the following term, plus a constant.

Let S_x denote the sum of the x first terms of a series whose general term is u_x,

$$\text{then } S_x = u_1 + u_2 + \dots + u_x,$$

$$S_{x+1} = u_1 + u_2 + \dots + u_x + u_{x+1};$$

$$\therefore S_{x+1} - S_x = \Delta S_x = u_{x+1};$$

$$\therefore S_x = \Sigma u_{x+1} + C.$$

$$\text{Making } x = 0, \quad S_0 = 0 = \Sigma u_{x=1} + C;$$

or, as it is usually written,

$$0 = \Sigma u_1 + C,$$

$$\therefore S_x = \Sigma u_{x+1} - \Sigma u_1.$$

In general the arbitrary constant will be determined by the term with which we make the series commence. Thus,

if we make it begin with u_r instead of u_1, this amounts to suppose the sum of the terms preceding u_r, that is, Σu_r to be zero; and therefore to determine the constant, we have

$$S_{r-1} = \Sigma u_{x=r} + C = 0.$$

89. As the summation of series, where the general term is given explicitly as a function of the index, thus resolves itself into the cases of integration treated of in Section II., it will only be necessary to give a few numerical examples; every expression integrated in that Section gives the sum of a series of which it is the general term.

1. To find the sum of the first x terms of any progression of figurate numbers. In the r^{th} order the general term is

$$u_x = \frac{x(x+1)(x+2)\dots(x+r-2)}{1.2.3\dots(r-1)};$$

$$\therefore\ \Sigma u_{x+1} = \frac{x(x+1)\dots(x+r-1)}{1.2.3\dots r} + C, \quad \text{(Art. 45.)}$$

$$\Sigma u_1 = 0 + C = 0;$$

$$\therefore\ S_x = \frac{x(x+1)\dots(x+r-1)}{1.2.3\dots r}.$$

Similarly, for the series of inverse figurate numbers, except when $r=2$, that is, for the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{x}$.

For we have $u_x = \dfrac{1.2.3\dots(r-1)}{x(x+1)\dots(x+r-2)}$;

$$\therefore\ \Sigma u_{x+1} = C - \frac{1.2.3\dots(r-1)}{(r-2)(x+1)\dots(x+r-2)}, \quad \text{(Art. 47.)}$$

$$\Sigma u_1 = C - \frac{r-1}{r-2} = 0,$$

$$S_x = (r-1)\left\{\frac{1}{r-2} - \frac{1.2.3\ldots(r-3)}{(x+1)(x+2)\ldots(x+r-2)}\right\},$$

and $S_\infty = \dfrac{r-1}{r-2}$.

2. To find the sums of the squares and cubes of the natural numbers,

$$u_x = x^2 = (x-1)x + x,$$

$$\therefore\ \Sigma u_{x+1} = \frac{(x-1)x(x+1)}{3} + \frac{x(x+1)}{2} + C,$$

$$\Sigma u_1 = 0 + C = 0;$$

$$\therefore\ S_x = \frac{x(x+1)(2x+1)}{6}.$$

Again, $u_x = x^3 = x(x^2-1) + x$;

$$\therefore\ \Sigma u_{x+1} = \frac{(x-1)x(x+1)(x+2)}{4} + \frac{x(x+1)}{2} + C,$$

$$\Sigma u_1 = 0 + C = 0;$$

$$\therefore\ S_x = \frac{x(x+1)}{2}\left\{\frac{(x-1)(x+2)}{2} + 1\right\} = \left\{\frac{x(x+1)}{2}\right\}^2.$$

Similarly, $1^2 + 3^2 + 5^2 + \ldots + (2x-1)^2 = \dfrac{x(4x^2-1)}{3}$.

$$1^3 + 3^3 + 5^3 + \ldots + (2x-1)^3 = 2x^4 - x^2.$$

3. To sum the series

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots \pm x^2;$$

$u_x = (-1)^{x-1}x^2$;

$$\Sigma u_x = \frac{(-1)^{x-1}x^2}{-2} - \frac{(2x+1)(-1)^x}{4} + \frac{2(-1)^{x+1}}{-8} + C.\ \text{(Art. 59.)}$$

$$\therefore \quad \Sigma u_{x+1} = (-1)^{x+1}\left\{\frac{(x+1)^2}{2} - \frac{2x+3}{4} + \frac{1}{4}\right\} + C.$$

$$\therefore \quad S_x = (-1)^{x+1}\frac{x(x+1)}{2}.$$

Similarly,

$$2.1^2 + 2^2.2^2 + 2^3.3^2 + \ldots + 2^x x^2 = 2^{x+1}(x+1)(x-3) + (2^{x+1}-1)\,6.$$

4. To find the sum of x terms of the series

$$\frac{1}{2^2-3^2} + \frac{1}{4^2-3^2} + \frac{1}{6^2-3^2} + \&c.$$

$u_x = \dfrac{1}{(2x)^2-3^2} = \dfrac{1}{(2x-3)(2x+3)}$, which falls under Art. 48.

$$S_x = \frac{1}{18} - \frac{x+1}{(2x+1)(2x+3)} - \frac{1}{6}\cdot\frac{6x+5}{(2x-1)(2x+1)(2x+3)};$$

$$\text{and} \quad S_\infty = \frac{1}{18}.$$

Similarly,

$$\frac{5}{1.2.3} + \frac{8}{2.3.4} + \frac{11}{3.4.5} + \&c. \text{ to } x \text{ terms} = 2 - \frac{3x+4}{(x+1)(x+2)}.$$

5. $2.4.7 + 4.7.13 + 8.13.25 + 16.25.49 + \&c.$ to x terms.

$$u_x = 2^x(3.2^{x-1}+1)(3.2^x+1).$$

Assume $\Sigma u_{x+1} = A(3.2^{x-1}+1)(3.2^x+1)(3.2^{x+1}+1)$; (Art. 55.)

$$\therefore \quad u_{x+1} = A(3.2^x+1)(3.2^{x+1}+1)\left\{3.2^{x+2}+1-(3.2^{x-1}+1)\right\}$$

$$= \frac{21}{4}A.2^{x+1}(3.2^x+1)(3.2^{x+1}+1);$$

$$\therefore \quad A = \frac{4}{21},$$

$$\therefore \quad \Sigma u_{x+1} = \frac{4}{21}(3 \,.\, 2^{x-1}+1)(3 \,.\, 2^{x}+1)(3 \,.\, 2^{x+1}+1)+C;$$

$$\Sigma u_1 = \frac{4}{21} \,.\, \frac{5}{2} \,.\, 4 \,.\, 7 + C = 0;$$

$$\therefore \quad S_x = \frac{4}{21}(3 \,.\, 2^{x-1}+1)(3 \,.\, 2^{x}+1)(3 \,.\, 2^{x+1}+1) - \frac{40}{3}.$$

6. $\dfrac{2}{3 \,.\, 3} - \dfrac{4}{3 \,.\, 9} + \dfrac{8}{9 \,.\, 15} - \dfrac{16}{15 \,.\, 33} + \text{\&c.}$

$$u_x = \frac{(-2)^x}{\{(-2)^x - 1\}\{(-2)^{x+1} - 1\}};$$

$$\therefore \quad S_x = \frac{1}{9} + \frac{1}{3} \,.\, \frac{1}{(-2)^{x+1} - 1}, \text{ (Art. 55.), and } S_\infty = \frac{1}{9}.$$

7. $\dfrac{16}{2 \,.\, 3 \,.\, 4} - \dfrac{21}{3 \,.\, 4 \,.\, 5} \,.\, \dfrac{2}{3} + \dfrac{26}{4 \,.\, 5 \,.\, 6} \,.\, \left(\dfrac{2}{3}\right)^2 - \text{\&c. to } x \text{ terms,}$

$$u_x = \frac{16 + 5(x-1)}{(x+1)(x+2)(x+3)} \left(-\tfrac{2}{3}\right)^{x-1}.$$

Assume $\quad \Sigma u_{x+1} = \dfrac{A}{(x+2)(x+3)} \left(-\tfrac{2}{3}\right)^x$, (Art. 56.)

then $\quad A = -3$, and

$$\therefore \quad S_x = \tfrac{1}{2} - \frac{3}{(x+2)(x+1)} \left(-\tfrac{2}{3}\right)^x.$$

Similarly,

$$\frac{19}{1 \,.\, 2 \,.\, 3} \,.\, \frac{1}{4} + \frac{28}{2 \,.\, 3 \,.\, 4} \,.\, \frac{1}{8} + \frac{39}{3 \,.\, 4 \,.\, 5} \,.\, \frac{1}{16} + \frac{52}{4 \,.\, 5 \,.\, 6} \,.\, \frac{1}{32} + \dots$$

to x terms,

$$u_x = \frac{x^2 + 6x + 12}{x(x+1)(x+2)} \left(\tfrac{1}{2}\right)^{x+1},$$

$$S_x = 1 - \frac{x+4}{(x+1)(x+2)} \left(\tfrac{1}{2}\right)^{x+1}.$$

8. $$\frac{1}{\cos\theta \cos 2\theta} + \frac{1}{\cos 2\theta \cos 3\theta} + \frac{1}{\cos 3\theta \cos 4\theta} + \&c.$$

to x terms.

$$u_x = \frac{1}{\cos x\theta \cos (x+1)\theta},$$

$$\Sigma u_{x+1} = \frac{\tan (x+1)\theta}{\sin\theta} + C, \quad \text{(Art. 53)}.$$

$$\Sigma u_1 = \frac{\tan\theta}{\sin\theta} + C = 0,$$

$$\therefore \quad S_x = \frac{\tan (x+1)\theta - \tan\theta}{\sin\theta}.$$

9. $1\cos\theta + 2\cos 2\theta + 3\cos 3\theta + \&c.$, to x terms.

$$u_{x+1} = (x+1)\cos (x+1)\theta,$$

$$\Sigma u_{x+1} = (x+1)\frac{\sin (x+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} - \frac{1}{2\sin\frac{\theta}{2}}\Sigma \sin (x+\tfrac{3}{2})\theta \text{ (Art. 59.)}$$

$$= \frac{(x+1)\sin (x+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} + \frac{\cos (x+1)\theta}{\left(2\sin\frac{\theta}{2}\right)^2} + C,$$

$$\Sigma u_1 = \qquad \tfrac{1}{2} \qquad + \frac{\cos\theta}{\left(2\sin\frac{\theta}{2}\right)^2} + C = 0,$$

$$\therefore \quad C = -\frac{1}{\left(2\sin\frac{\theta}{2}\right)^2},$$

$$\therefore \quad S_x = \frac{x \sin(x+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} + \frac{1}{2\sin^2\frac{\theta}{2}}\left\{\sin(x+\tfrac{1}{2})\theta \sin\frac{\theta}{2} - \sin^2\frac{x+1}{2}\theta\right\}$$

$$= \frac{x \sin(x+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} - \frac{1}{2}\left(\frac{\sin\frac{x\theta}{2}}{\sin\frac{\theta}{2}}\right)^2.$$

Hence also if $S_x = 1\cos^2\theta + 2\cos^2 2\theta + 3\cos^2 3\theta + \&c.$

then $\quad 2S_x = 1(1+\cos 2\theta) + 2(1+\cos 4\theta) + \&c.$

$$\therefore \quad S_x = \frac{x(x+1)}{4} + \frac{x\sin(2x+1)\theta}{4\sin\theta} - \frac{1}{4}\left(\frac{\sin x\theta}{\sin\theta}\right)^2.$$

10. $\tan^{-1}\dfrac{1}{1+1+1^2} + \tan^{-1}\dfrac{1}{1+2+2^2} + \tan^{-1}\dfrac{1}{1+3+3^2} + \&c.$

to x terms.

$$u_x = \tan^{-1}\frac{1}{1+x+x^2} = \tan^{-1}\frac{\Delta x}{1+x(x+1)},$$

$$\therefore \quad \Sigma u_x = \tan^{-1} x + C, \quad \text{(Art. 54.)}$$

$$\therefore \quad S_x = \Sigma u_{x+1} - \Sigma u_1 = \tan^{-1}(x+1) - \frac{\pi}{4}.$$

Recurring Series.

90. We next come to the case where the general term is not given explicitly in terms of its index, but only certain relations between the consecutive terms, or these and their indices are expressed; of this sort of series the most remarkable are Recurring Series.

A recurring series is a series in which an equation of the first degree with constant coefficients, holds good between a certain definite number of consecutive terms, in whatever part of the series they be taken.

For example, in the series

$$3 + 5 + 9 + 17 + 33 + \&c.$$

we have $9 = 3.5 - 2.3$, $17 = 3.9 - 2.5$, &c.; and in general

$$u_{x+2} = 3\,u_{x+1} - 2\,u_x.$$

The general equation of every recurring series is

$$u_{x+n} + p_1\,u_{x+n-1} + p_2\,u_{x+n-2} + \ldots + p_{n-1}\,u_{x+1} + p_n\,u_x = 0.$$

The series of coefficients which connects any term with the preceding ones is called the Scale of Relation. Thus

$$1 + p_1 + p_2 + \ldots + p_n = 0$$

is the scale of relation of the recurring series whose equation is

$$u_{x+n} + p_1\,u_{x+n-1} + \ldots + p_n\,u_x = 0. \qquad (1)$$

91. A recurring series may generally be resolved into two or more geometric progressions.

For if $a_1, a_2, a_3, \ldots a_n$ be the roots of the equation

$$a^n + p_1\,a^{n-1} + p_2\,a^{n-2} + \ldots + p_n = 0, \qquad (2)$$

the complete integral of the equation of the series is

$$u_x = c_1\,a_1^x + c_2\,a_2^x + \ldots + c_n\,a_n^x, \quad \text{(Art. 78.)}$$

$$\therefore \quad u_1 = c_1\,a_1 + c_2\,a_2 + c_3\,a_3 + \ldots + c_n\,a_n,$$

$$u_2 = c_1\,a_1^2 + c_2\,a_2^2 + c_3\,a_3^2 + \ldots + c_n\,a_n^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_n = c_1\,a_1^n + c_2\,a_2^n + c_3\,a_3^n + \ldots + c_n\,a_n^n,$$

and the series consequently is transformed into

$$c_1\,(a_1 + a_1^2 + \ldots + a_1^n) + c_2\,(a_2 + a_2^2 + \ldots + a_2^n) + \&c.$$
$$+ c_n\,(a_n + a_n^2 + \ldots + a_n^n).$$

92. In the particular cases in which equation (2) has equal, or impossible roots, the recurring series can no longer be resolved into geometric progressions; for the complete integral of the equation of the series becomes in those two cases, respectively,

$$u_x = (c_0 + c_1 x + c_2 x^2 + \ldots + c_{r-1} x^{r-1})\, a_1^x + c_{r+1} a_{r+1}^x + \ldots + c_n a_n^x,$$

$$u_x = (c_0 + c_1 x + \ldots + c_{r-1} x^{r-1})\, \rho^x \cos x\theta$$

$$+ (c_0^1 + c_1^1 x + \ldots + c_{r-1}^1 x^{r-1})\, \rho^x \sin x\theta + c_{2r+1} a_{2r+1}^x + \ldots + c_n a_n^x.$$

93. Hence, to find the general term of a recurring series, we must integrate the equation expressing the relation between its successive terms, and determine the arbitrary constants by making the general term u_x coincide with a sufficient number of given terms. When a series is known to be recurring, its equation may be determined by assuming it to be of the form (1); and then forming a sufficient number of equations for finding the coefficients p_1, p_2, &c. p_n, by substituing the given terms in order.

94. To find the sum of x terms of a recurring series.

First, let its general term be of the form

$$u_x = c_1 a_1^x + c_2 a_2^x + \ldots + c_n a_n^x,$$

$$\therefore\ S_x = \Sigma u_{x+1} - \Sigma u_1 = c_1 \cdot \frac{a_1^{x+1} - a_1}{a_1 - 1} + c_2 \frac{a_2^{x+1} - a_2}{a_2 - 1} + \ldots$$

$$+ c_n \cdot \frac{a_n^{x+1} - a_n}{a_n - 1}.$$

Secondly, let the general term be

$$u_x = (c_0 + c_1 x + c_2 x^2 + \ldots + c_{r-1} x^{r-1})\, a_1^x + c_{r+1} a_{r+1}^x + \&c.;$$

then since

$$\Sigma x^n a^x = \frac{a^x x^n}{a-1} - \frac{a^{x+1} \Delta x^n}{(a-1)^2} + \frac{a^{x+2} \Delta^2 x^n}{(a-1)^3} - \&c. \quad \text{(Art. 59.)}$$

$$\Sigma u_{x+1} = c_0 \frac{a_1^{x+1}}{a_1 - 1} + c_1 \left\{ \frac{a_1^{x+1}(x+1)}{a_1 - 1} - \frac{a_1^{x+2}}{(a_1 - 1)^2} \right\} + \&c.;$$

$\therefore\ S_x = \Sigma u_{x+1} - \Sigma u_1$, is determined.

Thirdly, let

$$u_x = \rho^x (c_0 \cos x\theta + c_0' \sin x\theta) + \rho^x x (c_1 \cos x\theta + c_1' \cos x\theta)$$
$$+ \rho^x x^2 (c_2 \cos x\theta + c_2' \sin x\theta) + \&c.,$$

then each term to be integrated will be of the form

$$x^n \rho^x \cos (x\theta + a),$$

the integral of which may be found by Art. 58, because

$$\Sigma^n \{\rho^x \cos (x\theta + a)\}$$

is always assignable.

Ex. 1. To find the sum of x terms of the series

$$1 + 5 + 17 + 53 + \&c.$$

Let the equation be $u_{x+2} + pu_{x+1} + qu_x = 0$;

$$\therefore\ 17 + 5p + q = 0, \quad 53 + 17p + 5q = 0,$$

which give $p = -4, \quad q = 3$;

$$\therefore\ u_{x+2} - 4u_{x+1} + 3u_x = 0.$$

Let $u_x = a^x$; $\quad \therefore\ a^2 - 4a + 3 = 0$; $\quad \therefore\ a = 3$ or 1;

$$\therefore\ u_x = c_1 3^x + c_2,$$

$$1 = 3c_1 + c_2, \quad 5 = 9c_1 + c_2,$$

$$\therefore\ c_1 = \tfrac{2}{3}, \quad c_2 = -1,$$

$$\therefore\ u_x = 2 \,.\, 3^{x-1} - 1;$$

$$\therefore\ S_x = 2 \,.\, \frac{3^x}{3 - 1} - x + C, \quad 0 = 1 + C;$$

$$\therefore\ S_x = 3^x - x - 1.$$

Ex. 2. $\quad 2 - a - a^2 + 2a^3 - a^4 - a^5 + 2a^6 - a^7 - \&c.$

$$u_x = 2a^{x-1} \cos \frac{(2x - 2)\pi}{3}.$$

Ex. 3. $1 + 4 + 18 + 80 + 356 +$ &c., to x terms.

Here $u_{x+2} - 4u_{x+1} - 2u_x = 0$;

$$\therefore\ 2\sqrt{6}\,u_x = (2+\sqrt{6})^x - (2-\sqrt{6})^x,$$

and

$$2\sqrt{6}\,S_x = \frac{(2+\sqrt{6})^{x+1} - (2+\sqrt{6})}{1+\sqrt{6}} - \frac{(2-\sqrt{6})^{x+1} - (2-\sqrt{6})}{1-\sqrt{6}}.$$

Application of the Integral Calculus to the Summation of Series.

95. As integrals are often expressed by Series, so, conversely, the latter may be represented by integrals; and it is often desirable to find the integral of which a proposed series is one of the developments, in order to subject it to the methods which we possess for calculating, at least approximately, the value of any integral taken between assigned limits. We proceed therefore to notice one or two processes given by Euler for effecting this; they consist chiefly in performing certain operations on the series, by which it is transformed into another series which we are able to sum, or which is similar to the proposed one.

96. Series which proceed according to the powers of some quantity t, affected with coefficients consisting of factors in arithmetic progression either in the numerator or denominator, may be summed by the aid of the Integral Calculus, the denominators being taken away by differentiation and the numerators by integration.

Ex. 1. Let $s = \frac{t}{2} + \frac{2t^2}{3} + \frac{3t^3}{4} +$ &c. (to ∞),

$$\therefore\ d_t(st) = t + 2t^2 + 3t^3 + \text{\&c.}$$

$$\therefore\ \int_t \frac{d_t(st)}{t} = t + t^2 + t^3 + \text{\&c.} = \frac{t}{1-t};$$

$$\therefore \frac{d_t(st)}{t} = \frac{1}{(1-t)^2},$$

$$\therefore st = \int_t \frac{t}{(1-t)^2} = \log(1-t) + \frac{1}{1-t} + C,$$

$$0 \qquad = \qquad 0 \quad + 1 + C;$$

$$\therefore s = \frac{\log(1-t)}{t} + \frac{1}{1-t}.$$

Ex. 2. $$s = \frac{a+b}{a+\beta}t + \frac{a+2b}{a+2\beta}t^2 + \dots$$

$$+ \frac{a+nb}{a+n\beta}t^n + \dots \text{ to } m \text{ terms.}$$

$$\therefore \beta d_t(st^{\frac{a}{\beta}}) = \dots + (a+nb)\, t^{n+\frac{a}{\beta}-1} + \dots$$

$$\frac{\beta}{b}\int_t\left\{t^{\frac{a}{b}-\frac{a}{\beta}} d_t(st^{\frac{a}{\beta}})\right\} = \dots + t^{n+\frac{a}{b}} + \dots = \frac{t^m-1}{t-1}\, t^{\frac{a}{b}+1};$$

$$\therefore s = \frac{b}{\beta}t^{-\frac{a}{\beta}}\int_t t^{\frac{a}{\beta}-\frac{a}{b}} . d_t\left\{\frac{t^m-1}{t-1} . t^{\frac{a}{b}+1}\right\}.$$

Ex. 3. $$s = 1 + \frac{a}{a+\beta}t + \frac{a(a+b)}{(a+\beta)(a+2\beta)}t^2$$

$$+ \frac{a(a+b)(a+2b)}{(a+\beta)(a+2\beta)(a+3\beta)}t^3 + \text{\&c. (to } \infty),$$

$$\frac{1}{b}\int_t st^{\frac{a}{b}-2} = \frac{t^{\frac{a}{b}-1}}{a-b} + \frac{t^{\frac{a}{b}}}{a+\beta} + \frac{a}{(a+\beta)(a+2\beta)}t^{\frac{a}{b}+1}$$

$$+ \frac{a(a+b)}{(a+\beta)(a+2\beta)(a+3\beta)}t^{\frac{a}{b}+2} + \text{\&c.};$$

$$\therefore\ \frac{\beta}{b} d_t\left(t^{\frac{\alpha}{\beta}\quad \frac{a}{b}+1} . \int_t s\, t^{\frac{a}{b}-2}\right) = \frac{a}{a-b} t^{\frac{\alpha}{\beta}-1} + t^{\frac{\alpha}{\beta}} + \frac{a}{\alpha+\beta} t^{\frac{\alpha}{\beta}+1}$$

$$+ \frac{a(a+b)}{(\alpha+\beta)(\alpha+2\beta)} t^{\frac{\alpha}{\beta}+2} + \text{\&c.} \quad (1.)$$

$$= \frac{a}{a-b} t^{\frac{\alpha}{\beta}-1} + t^{\frac{\alpha}{\beta}} s;$$

hence, performing the differentiation, we find

$$(\beta t - bt^2)\, d_t s + (\alpha - at)\, s = a,$$

a linear equation of the first order for finding s.

For the sum of m terms, we shall evidently have from equation (1.)

$$\frac{\beta}{b} d_t\left(t^{\frac{\alpha}{\beta}-\frac{a}{b}+1} \int_t s\, t^{\frac{a}{b}-2}\right) = \frac{a}{a-b} t^{\frac{\alpha}{\beta}-1}$$

$$+ t^{\frac{\alpha}{\beta}} \left\{ s - \frac{a(a+b) \ldots \{a+(m-2)b\}}{(\alpha+\beta)(\alpha+2\beta) \ldots \{\alpha+(m-1)\beta\}} t^{m-1} \right\}.$$

Ex. 4. $$s = 1 + \frac{\alpha}{1} . \frac{\beta}{\gamma} t + \frac{\alpha(\alpha+1)}{1.2} \frac{\beta(\beta+1)}{\gamma(\gamma+1)} t^2$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2)}{1.2.3} \frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)} t^3 + \text{\&c. (to } \infty).$$

$$(t^2 - t)\, d_t^2 s + \{(\alpha+\beta+1)\, t - \gamma\}\, d_t s + \alpha\beta s = 0.$$

97. When consecutive denominators have only a single factor in common, or none at all, the following is a convenient mode of summing the series.

Ex. $$\frac{1}{a(a+b)(a+2b)} + \frac{1}{(a+2b)(a+3b)(a+4b)}$$

$$+ \text{\&c. (to } \infty).$$

Since $\dfrac{2b^2}{(a+xb)\{a+(x+1)b\}\{a+(x+2)b\}} = \dfrac{1}{a+xb}$

$$- \frac{2}{a+(x+1)b} + \frac{1}{a+(x+2)b},$$

resolving all the terms by means of this formula, by putting $x = 0, 1, 2$, &c., we find

$$2b^2 S = \frac{1}{a} - \frac{2}{a+b} + \frac{2}{a+2b} - \frac{2}{a+3b} + \frac{2}{a+4b} - \text{\&c.}$$

$$\therefore\ 2b^2 S = 2\int_x^{0\ 1} \frac{x^{a-1}}{1+x^b} - \frac{1}{a}.$$

Similarly, $\dfrac{1}{a(a+b)(a+2b)(a+3b)}$

$$+ \frac{1}{(a+3b)(a+4b)(a+5b)(a+6b)} + \text{\&c. (to } \infty).$$

$$= \frac{1}{6b^3}\left(\frac{1}{a} - 3\int_x^{0\ 1} \frac{x^{a+b-1}}{1+x^b+x^{2b}}\right).$$

98. We may obtain an expression for the sum of the series,

$$1 + \frac{x}{1} \cdot d_z\phi(z) + \frac{x^2}{1.2} d_z^2\{\phi(z)\}^2 + \frac{x^3}{1.2.3} d_z^3\{\phi(z)\}^3 + \text{\&c.},$$

where $\phi(z)$ is any function of z, by Lagrange's Theorem.

For if $y = z + x\phi(y)$, then

$$y = z + \frac{x}{1}\phi(z) + \frac{x^2}{1.2} d_z\{\phi(z)\}^2 + \text{\&c.}$$

$$\therefore\ 1 + \frac{x}{1} d_z\phi(z) + \frac{x^2}{1.2} d_z^2\{\phi(z)\}^2 + \text{\&c.} = d_z y = \frac{1}{1 - x d_y\phi(y)},$$

where $d_y\phi(y)$ must be obtained in terms of x and z, from the equation $y = z + x\phi(y)$. Suppose, for example,

$\phi(y) = \frac{1}{2}(y^2 - 1)$, then $d_y\phi(y) = y$,

and $y = z + \frac{x}{2}(y^2 - 1)$, $\therefore\ 1 - xy = \sqrt{1 - 2zx + x^2}$;

$$\therefore\ 1 + \frac{x}{1}\frac{1}{2}d_z(z^2-1) + \frac{x^2}{1.2}\cdot\frac{1}{2^2}\cdot d_z^2(z^2-1)^2 + \ldots = \frac{1}{\sqrt{1-2zx+x^2}}.$$

Hence if $(1 - 2zx + x^2)^{-\frac{1}{2}} = 1 + Z_1 x + Z_2 x^2 + \ldots + Z_n x^n + \ldots$,

$$\text{then } Z_n = \frac{1}{2^n \lfloor n} d_z^n (z^2 - 1)^n.$$

The function Z_n has some remarkable properties, of which the following may be noticed.

From the formula

$$\int_x \frac{1}{\sqrt{(a+bx)(c+ex)}} = \frac{2}{\sqrt{eb}} \log\left\{\sqrt{a+bx} + \sqrt{\frac{b}{e}(c+ex)}\right\}$$

it is easily shewn that

$$\int_z^{-1,+1} (1 - 2arz + a^2r^2)^{-\frac{1}{2}}\cdot\left(1 - \frac{2a}{r}z + \frac{a^2}{r^2}\right)^{-\frac{1}{2}} = \frac{1}{a}\log\left(\frac{1+a}{1-a}\right),$$

$$\therefore\quad \int_z^{-1,+1} (1 + Z_1 ar + Z_2 a^2r^2 + \ldots)\cdot\left(1 + Z_1\frac{a}{r} + Z_2\frac{a^2}{r^2} + \ldots\right)$$

$$= \frac{2}{a}\left(a + \frac{a^3}{3} + \frac{a^5}{5} + \&c.\right).$$

Hence the definite integral of every term of the product forming the first member that is not independent of r, must equal zero;

$$\therefore\quad \int_z^{-1,+1} (Z_n Z_m) = 0, \quad\text{and}\quad \int_z^{-1,+1} (Z_n)^2 = \frac{2}{2n+1}.$$

99. To find an expression for the sum of the reciprocals of the n^{th} powers of the values of y, in the equation

$$z - y + \phi(y) = 0;$$

where $\phi(y)$ denotes any function of y.

Let $z - y + \phi(y) = C(a_1 - y)(a_2 - y) \ldots (a_m - y)$,

$$\therefore \quad \frac{1 - d_y \phi(y)}{z - y + \phi(y)} = \frac{1}{a_1 - y} + \frac{1}{a_2 - y} + \ldots + \frac{1}{a_m - y}. \qquad (1.)$$

Now developing the two members of this equation in powers of y, the general term of the second member is

$$\left(\frac{1}{a_1^{n+1}} + \frac{1}{a_2^{n+1}} + \ldots + \frac{1}{a_m^{n+1}}\right) y^n = s_{-n-1} \cdot y^n,$$

and by Taylor's theorem the general term of the first member, putting $1 - d_y \phi(y) = f(y)$, is

$$\frac{f(y)\,\phi^r(y)}{\lfloor r} \, d_z^r \left(\frac{1}{z - y}\right).$$

But if $f(y) = A_0 + A_1 y + \ldots + A_n y^n + \ldots$,

$$\text{since } \frac{1}{z - y} = \frac{1}{z} + \frac{y}{z^2} + \ldots + \frac{y^n}{z^{n+1}} + \ldots,$$

$$\therefore \quad \frac{f(y)}{z - y} = \ldots + \frac{A_0 + A_1 z + \ldots + A_n z^n}{z^{n+1}} y^n + \ldots,$$

therefore the general term of $\dfrac{f(y)}{z - y}$ is $\dfrac{f(z)}{z^{n+1}} y^n$, whatever be the form of $f(z)$, if $\dfrac{f(z)}{z^{n+1}}$ be restricted to such terms as involve negative powers of z only; consequently the general term of $\dfrac{f(y)\,\phi^r(y)}{z - y}$ is $\dfrac{f(z)\,\phi^r(z)}{z^{n+1}} \cdot y^n$, and therefore of

$$\frac{1}{\lfloor r} \, d_z^r \left(\frac{f(y)\,\phi^r(y)}{z - y}\right) \text{ it is } \frac{1}{\lfloor r} \, d_z^r \left(\frac{f(z)\,\phi^r(z)}{z^{n+1}}\right) y^n;$$

hence equating coefficients of y^n on both sides of equation (1),

$$s_{-n-1} = \frac{f(z)}{z^{n+1}} + d_z \left(\frac{f(z)\,\phi(z)}{z^{n+1}}\right) + \frac{1}{1 \cdot 2} d_z^2 \left(\frac{f(z)\,\phi^2(z)}{z^{n+1}}\right) + \&c.$$

Now putting for $f(z)$ its value, the general term may be resolved into

$$\frac{1}{\lfloor r} d_z^r \left(\frac{\phi^r(z)}{z^{n+1}}\right) - \frac{1}{\lfloor r} d_z^r \left(\frac{\phi'(z) \cdot \phi^r(z)}{z^{n+1}}\right),$$

$$\text{or } \frac{1}{\lfloor r-1} d_z^{r-1} \left(\frac{\phi^{r-1}(z)\,\phi'(z)}{z^{n+1}}\right) - \frac{n+1}{\lfloor r} d_z^{r-1} \left(\frac{\phi^r(z)}{z^{n+2}}\right)$$
$$- \frac{1}{\lfloor r} d_z^r \left(\frac{\phi^r(z)\,\phi'(z)}{z^{n+1}}\right),$$

of which the first and last will be destroyed by the corresponding parts in the preceding and succeeding terms; therefore, changing n into $n-1$,

$$s_{-n} = \frac{1}{z^n} - \frac{n}{1}\frac{\phi(z)}{z^{n+1}} - \frac{n}{1\,.\,2} d_z \left(\frac{\phi^2(z)}{z^{n+2}}\right) - \ldots - \frac{n}{\lfloor r} d_z^{r-1} \left(\frac{\phi^r(z)}{z^{n+2}}\right) - \&c.$$

But the second member of this equation is what Lagrange's Theorem gives for the development of y^{-n}, in the equation $z - y + \phi(y) = 0$; therefore s_{-n} is equal to the sum of those terms of the development of y^{-n} which involve negative powers of z.

100. By the help of the preceding proposition it may be shewn that Lagrange's Theorem, applied to the solution of the equation

$$z - y + \phi(y) = 0,$$

gives an approximation to the least root. By what precedes, we have

$$\frac{s_{-n}}{s_{-n-r}} = \frac{\dfrac{1}{z^n} - \dfrac{n}{1}\dfrac{\phi(z)}{z^{n+1}} - \dfrac{n}{1\,.\,2} d_z \left(\dfrac{\phi^2(z)}{z^{n+1}}\right) - \&c.}{\dfrac{1}{z^{n+r}} - \dfrac{n+r}{1}\dfrac{\phi(z)}{z^{n+r+1}} - \dfrac{n+r}{1\,.\,2} d_z \left(\dfrac{\phi^2(z)}{z^{n+r+1}}\right) - \&c.},$$

where each series is restricted to those terms which involve negative powers of z; and the number of those terms increases with (n), and if n be supposed very great, then each

series may be taken ad infinitum. But whatever be the value of n, if both series go on ad infinitum, the value of the second member is

$$\frac{y^{-n}}{y^{-n-r}} = y^r = z^r + \frac{r}{1} z^{r-1} \phi(z) + \frac{r}{1.2} d_z \{z^{r-1} \phi(z)\} + \&c.,$$

as given by Lagrange's Theorem, and which may be verified by actual multiplication; therefore when n is infinite,

$$\text{limit of } \frac{s_{-n}}{s_{-n-r}} = y^r,$$

(y^r denoting that function of z which is given for the developement of y^r by Lagrange's Theorem).

But if a_1 be the least of the roots a_1, a_2, a_3, &c., we have also (Theory of Algebraical Equations, Art. 156.)

$$\text{limit of } \frac{s_{-n}}{s_{-n-r}} = a_1^r, \text{ when } n \text{ is infinite;}$$

$$\therefore \quad y^r = a_1^r, \quad \text{and} \quad y = a_1, \text{ the least root.}$$

101. To find the sums of the series

$$\frac{\sin a}{1^2 + k^2} + \frac{2 \sin 2a}{2^2 + k^2} + \frac{3 \sin 3a}{3^2 + k^2} + \&c. \text{ (to } \infty),$$

$$\frac{k \cos a}{1^2 + k^2} + \frac{k \cos 2a}{2^2 + k^2} + \frac{k \cos 3a}{3^2 + k^2} + \&c. \text{ (to } \infty),$$

or the values of $\quad {}_1S^\infty \dfrac{n \sin na}{n^2 + k^2}, \quad {}_1S^\infty \dfrac{k \cos na}{n^2 + k^2}.$

First, we have

$$\frac{(1-p^2)\, e^{-kx}}{1 - 2p\cos x + p^2} = e^{-kx} + 2\,(p \cos x + p^2 \cos 2x + p^3 \cos 3x + \&c.)\, e^{-kx},$$

and integrating both sides from $x = 0$ to $x = a$, and putting $p = 1$, we get

$$\pi = \frac{1 - e^{-ka}}{k} + 2e^{-ka}\,{}_1S^{\infty}\,\frac{n \sin na}{k^2 + n^2} - 2e^{-ka}\,{}_1S^{\infty}\,\frac{k \cos na}{k^2 + n^2}$$

$$+ 2\,{}_1S^{\infty}\,\frac{k}{k^2 + n^2}.$$

Again, integrating from $x = 0$ to $x = 2\pi$, and putting $p = 1$, we find

$$\pi(e^{-2k\pi} + 1) = \frac{1 - e^{-2k\pi}}{k} + 2\,{}_1S^{\infty}\,\frac{k}{k^2 + n^2} - 2e^{-2k\pi}\,{}_1S^{\infty}\,\frac{k}{k^2 + n^2}.$$

Now subtract the foregoing result from this, and call the two sums we are in search of, S_1 and S_2, then

$$\frac{2\pi e^{-2k\pi + ka}}{1 - e^{-2k\pi}} = \frac{1}{k} - 2S_1 + 2kS_2;$$

and changing the sign of k,

$$\frac{-2\pi e^{-ka}}{1 - e^{-2k\pi}} = -\frac{1}{k} - 2S_1 - 2kS_2;$$

therefore, adding and subtracting,

$$S_1 = {}_1S^{\infty}\,\frac{n \sin na}{k^2 + n^2} = \frac{\pi}{2}\,\frac{e^{k(\pi - a)} - e^{-k(\pi - a)}}{e^{k\pi} - e^{-k\pi}},$$

$$S_2 = {}_1S^{\infty}\,\frac{k \cos na}{k^2 + n^2} = \frac{\pi}{2k}\,\frac{e^{k(\pi - a)} + e^{-k(\pi - a)}}{e^{k\pi} - e^{-k\pi}} - \frac{1}{2k^2}.$$

These formulæ were first given by Poisson, and may be considered as embracing the chief results which have hitherto been obtained relating to the summation of series of the sines and cosines of multiple angles.

102. The definite integrals required in the preceding investigation may be found as follows. Suppose

$$y = \frac{(1-p^2)\, e^{-kx}}{1 - 2p \cos x + p^2},$$

to be the equation to a curve; and let it be proposed to find the limiting value of its area, from $x = 0$ to $x = a$, on the supposition that p approaches continually to unity. First, we observe that all the ordinates will be ultimately evanescent except those corresponding to $x = 0$, 2π, 4π, &c., which will be very large, because for those values $\frac{1+p}{1-p}$ becomes a factor of the expression for y. We will begin by supposing that $a < 2\pi$; and therefore the only ordinates that we are concerned with are those immediately succeeding that through the origin; so that, making $p = 1 - \rho$, and then supposing x very small, we get successively

$$y = \frac{\rho\,(2-\rho)\, e^{-kx}}{\rho^2 + 4\,(1-\rho)\sin^2 \frac{x}{2}} = \frac{2\rho}{\rho^2 + x^2}, \text{ ultimately;}$$

$$\therefore\ {}^0\!\!\int_x^a y = 2 \tan^{-} \frac{a}{\rho},$$

$${}^0\!\!\int_x^a y_{p=1} = \pi, \text{ making } \rho = 0.$$

Next, suppose that the area is to be found from $x = 0$ to $x = 2\pi$; then, besides the area just found, there will be another portion immediately preceding the point for which $x = 2\pi$; to find this latter portion, put $x = 2\pi - x'$, and call the ordinate y';

$$\text{then } y' = \frac{(1-p^2)\, e^{-2k\pi + kx'}}{1 - 2p \cos x' + p^2} = e^{-2k\pi} y,$$

therefore, the portion of the area immediately preceding the second limit $= \pi e^{-2k\pi}$;

$$\therefore\ {}^0\!\!\int_x^{2\pi} y_{p=1} = \pi + \pi e^{-2k\pi}.$$

103. As the following important theorems in Definite Integrals are based upon the preceding investigations, we shall here give the proofs of them, though not strictly belonging to the present subject.

$$y = \frac{(1-p^2)f(x)}{1-2p\cos(c-x)\frac{\pi}{a}+p^2} = f(x) + 2\,{}^1S^\infty p^n \cos n(c-x)\frac{\pi}{a}\cdot f(x);$$

$$\therefore\ {}^{-a}\!\int_x^{+a} y_{p=1} = {}^{-a}\!\int_x^{+a} f(x) + 2\,{}^1S^\infty\ {}^{-a}\!\int_x^{+a} \cos n(c-x)\frac{\pi}{a}\cdot f(x).$$

Now, suppose c to be less than a, then for all values of x between $x=-a$ and $x=+a$, y is evanescent, except when $x=c$; if therefore we write $x=c+z$, supposing z exceedingly small, and then integrate with respect to z, from $z=0$ to $z=\gamma$ any small finite value, and double the result, and make $p=1$, we shall obtain the value of ${}^{-a}\!\int_x^{+a} y_{p=1}$. But making $p=1-\rho$, we have

$$y = \frac{\rho(2-\rho)f(c+z)}{\rho^2+4(1-\rho)\sin^2\frac{z\pi}{2a}} = \frac{2\rho f(c)}{\rho^2+\frac{\pi^2 z^2}{a^2}} \text{ ultimately,}$$

$$= \frac{2af(c)}{\pi}\cdot\frac{\rho a\pi}{(\rho a)^2+(\pi z)^2};$$

$$\therefore\ {}^0\!\int_z^\gamma y = \frac{2af(c)}{\pi}\tan^{-1}\frac{\pi\gamma}{\rho a};$$

$$\therefore\ {}^0\!\int_z^\gamma y_{p=1} = af(c);$$

$$\therefore\ 2af(c) = {}^{-a}\!\int_x^{+a} f(x) + 2\,{}^1S^\infty\ {}^{-a}\!\int_x^{+a} \cos n(c-x)\frac{\pi}{a}\cdot f(x), \quad (1.)$$

$$\therefore\ f(c) = \frac{1}{2a}\,{}^{-a}\!\int_x^{+a} f(x) + \frac{1}{a}\,{}^{-a}\!\int_x^{+a}\Big\{\cos(c-x)\frac{\pi}{a}\cdot f(x)$$

$$+\cos(c-x)\frac{2\pi}{a}\cdot f(x) + \cos(c-x)\frac{3\pi}{a}\cdot f(x)+\dots\Big\};$$

therefore, making a infinite,

$$f(c)=\frac{1}{\pi}\text{ limit of }\frac{\pi}{a}\,{}^{-a}\!\!\int_x^{+a}\left\{\cos(c-x)\frac{\pi}{a}+\cos(c-x)\frac{2\pi}{a}+\&c.\right\}f(x),$$

$$\text{or, } f(c)=\frac{1}{\pi}\,{}^{0}\!\!\int_z^{\infty}\,{}^{-a}\!\!\int_x^{+a}\cos(c-x)z\,.f(x),\ \text{(Integ. Cal. Art. 116)},$$

a theorem given by Fourier, and included, as we see, in the theorem (1), which was first given by Poisson.

104. When in physical questions a definite integral arises whose value cannot be exhibited in finite terms, or in a form convenient for numerical calculation, the method of Quadratures is used as a substitute for Integration. This method consists in taking a series of values of the function to be integrated, multiplying these by the differences of the corresponding values of the independent variable, and adding together all the results. The sum of such results approximates to the value of the definite integral, as the intervals of the independent variable are diminished. The method is equivalent to adopting, for the area of a curve which a definite integral gives, the sum of the areas of a system of inscribed or circumscribed polygons.

Let the equal intervals into which the independent variable x is divided be taken equal to unity, and let $f(x)$ be the function of x to be integrated between assigned limits a and b. Take a value of it $f(a+n-\frac{1}{2})$ at the middle of the n^{th} of the intervals into which $b-a$ is divided. If the intervals be very small, $f(x)$ may be considered constant from

$$x=a+n-1 \quad\text{to}\quad x=a+n, \quad\text{or}\quad \Delta^2 f(x)$$

may be neglected; and the value of the definite integral is $\Sigma f(a+n-\frac{1}{2})$, where n is to receive in succession the values

$$1,\ 2,\ 3,\ldots b-a.$$

For greater accuracy, suppose $\Delta^2 f(x)$ to be constant from

$$x = a \quad \text{to} \quad x = b, \quad \text{or} \quad \Delta^3 f(x)$$

to be neglected. Then

$$f(a+n-\tfrac{1}{2}+z) = f(a+n-\tfrac{1}{2}) + z\,\Delta f(a+n-\tfrac{1}{2}) + \frac{z^2}{2}\,\Delta^2 f(x),$$

and the integral for the interval in question is

$$\textstyle\int_z f(a+n-\tfrac{1}{2}+z), \quad \text{from } z = -\tfrac{1}{2} \text{ to } z = \tfrac{1}{2},$$

$$= f(a+n-\tfrac{1}{2}) + \frac{1}{6}\left(\frac{1}{8}+\frac{1}{8}\right)\Delta^2 f(x) = f(a+n-\tfrac{1}{2}) + \frac{1}{24}\,\Delta^2 f(x).$$

The integral, consequently, from a to b, is

$$\Sigma \left\{ f(a+n-\tfrac{1}{2}) + \frac{1}{24}\,\Delta^2 f(x) \right\};$$

n receiving the same successive values as in the former case. In calculating the value of a definite integral by this method, the intervals are to be taken smaller as the variation of the function is more rapid.

Convergency and Divergency of Series.

105. A series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \ (\text{to } \infty)$$

is called convergent, if the sum s_n of any number n of its terms approaches continually to a finite quantity s as its limit, when n is indefinitely increased; and divergent in the contrary case.

106. When a series is convergent, the sum of any number of consecutive terms after the n^{th} continually tends to zero as n increases. For

$$u_{n+1} + u_{n+2} + \ldots + u_{n+m} = s_{n+m} - s_n;$$

therefore, as n increases, the value of the first member continually approaches to $s - s$, or zero, as its limit. This being true when $m = 1$, we see also that u_{n+1}, or the general term u_n continually tends to zero as n increases, or each term is greater than the following; but this, although a necessary condition, is not sufficient to insure the convergency of a series.

Thus in the series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n};$$

$u_n = \dfrac{1}{n}$ continually approaches to zero as n increases; but

$$s_{n+m} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+m}$$

is evidently greater than

$$\frac{m}{n+m} > \tfrac{1}{2}, \text{ if } m = n;$$

and therefore the sum of n consecutive terms after the n^{th}, does not diminish indefinitely as n increases; consequently, the series in question is divergent.

107. In the geometrical progression

$$a + ax + ax^2 + \ldots + ax^{n-1} + \ldots \quad (1.)$$

$$s_n = a \cdot \frac{1 - x^n}{1 - x},$$

$$s_{n+m} = a \cdot \frac{1 - x^{n+m}}{1 - x},$$

$$\therefore \; s_{n+m} - s_n = ax^n \cdot \frac{1 - x^m}{1 - x}.$$

Hence if $x < 1$, when n is infinite $x^n = 0$,

$$\therefore \quad s = \frac{a}{1-x};$$

$$\text{also} \quad u_n = 0, \quad s_{n+m} - s_n = 0;$$

both which results shew that the series is convergent.

But if $x > 1$, then $u_n = ax^{n-1}$ increases indefinitely with n, which alone shews that the series is divergent.

Hence the geometrical progression (1) is convergent or divergent, according as x is less or greater than unity; and it may be used as the test of the convergency or divergency of other series. For if a proposed series can be shewn to have no term greater than the corresponding term of (1) when $x < 1$, then that series is convergent; or if a proposed series can be shewn to have no term less than the corresponding term of (1) when $x > 1$, then that series is divergent.

Thus in the series for e, the base of the natural logarithms,

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots + \frac{1}{\underline{|n-1}} + \ldots$$

the terms which follow the n^{th}, viz.

$$\frac{1}{\underline{|n}} + \frac{1}{\underline{|n+1}} + \frac{1}{\underline{|n+2}} + \&c.$$

are evidently less than the corresponding terms of the geometrical progression

$$\frac{1}{\underline{|n}} + \frac{1}{\underline{|n}} \cdot \frac{1}{n} + \frac{1}{\underline{|n}} \cdot \frac{1}{n^2} + \&c.,$$

the sum of which is

$$\frac{1}{\underline{|n}} \cdot \frac{1}{1 - \frac{1}{n}} = \frac{1}{\underline{|n-1}} \cdot \frac{1}{n-1},$$

and continually tends to zero as n increases. Therefore the proposed series is convergent; and the error in taking the aggregate of its first n terms for its sum, is less than the quotient of the n^{th} term divided by $n - 1$.

108. From the measure of convergency or divergency which a geometrical progression furnishes, we shall now proceed to deduce one or two other tests as given by Cauchy, *Cours d'Analyse Algébrique.*

109. The series $u_1 + u_2 + \dots + u_n + \dots$ is convergent, or will become so, if the superior limit of $(u_n)^{\frac{1}{n}}$ be less than 1, when n is infinite; and divergent in the contrary case.

Let k denote the superior limit of $(u_n)^{\frac{1}{n}}$ when n is infinite; and first suppose $k < 1$; also, let a be any magnitude between k and 1, so that $k < a < 1$; then when n is increased indefinitely, $(u_n)^{\frac{1}{n}}$ cannot approach indefinitely near to k without finally becoming constantly less than a. Therefore it will be possible to take for n so large a value, that for that and all superior values, we may constantly have

$$(u_n)^{\frac{1}{n}} < a, \quad u_n < a^n.$$

Consequently, the proposed series will finish by always having its terms less than the corresponding terms of the geometrical progression

$$a + a^2 + \dots + a^n + a^{n+1} + \dots;$$

and as this series is convergent, a being < 1, it follows that the proposed series will end by being, *à fortiori*, convergent.

Secondly, suppose $k > 1$; and take, as before, a between 1 and k, so that $k > a > 1$. Then, when n is indefinitely increased, $(u_n)^{\frac{1}{n}}$ cannot approach indefinitely near to k without finally becoming constantly $> a$; we shall therefore be able

to satisfy the condition $(u_n)^{\frac{1}{n}} > a$, or $u_n > a^n$, by taking n sufficiently large; and consequently we shall always find in the series $u_1 + u_2 + \ldots + u_n + u_{n+1} + \ldots$, an indefinite number of terms greater than the corresponding terms of the geometrical progression $a + a^2 + \ldots a^n + a^{n+1} + \ldots$, which is divergent, a being > 1; and therefore the proposed series will end by being divergent.

110. It may be shewn that if, as n increases indefinitely, u_n remains positive, and the ratio $\frac{u_{n+1}}{u_n}$ continually tends to become equal to a finite quantity k as its limit, the expression $(u_n)^{\frac{1}{n}}$ continually approaches to the same limit. Hence, the test of convergency or divergency in the last Article, may be changed into the following, which is more convenient in its application; if the limit of the ratio of u_{n+1} to u_n when n is infinite be less than 1, the series is convergent; and divergent in the contrary case.

If $k = 1$, this test gives no result in either form.

111. Suppose that the series $u_1 + u_2 + u_3 + \ldots$ consists of both positive and negative terms; then if v_1, v_2, v_3, &c. be the numerical values of these terms, so that $u_1 = \pm v_1$, $u_2 = \pm v_2$, &c., it is evident that the sum of the proposed series can never surpass that of the series $v_1 + v_2 + v_3 + \ldots$; if therefore, the latter series be convergent, that is, if

$$\left(\frac{v_{n+1}}{v_n}\right)_{n=\infty} < 1,$$

the proposed series will be convergent; or, if the latter series finish by having terms greater than any assignable magnitude, that is, if

$$\left(\frac{v_{n+1}}{v_n}\right)_{n=\infty} > 1,$$

the same thing will happen to the proposed series, which will consequently be divergent. Hence, the above test is applicable to series consisting of both positive and negative terms, provided we use the numerical values of the terms without regard to signs; and it fails, as in the preceding case, when $k = 1$.

112. Let $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ (1.)

be a series arranged according to positive and ascending powers of the variable x; the coefficients being positive or negative; then, by what has been proved, this series will be convergent or divergent, according as kx, (where

$$k = (a_n)^{\frac{1}{n}} = \frac{a_{n+1}}{a_n},$$

when n is infinite,) is numerically less or greater than 1. Hence for all values of x between $-\frac{1}{k}$ and $+\frac{1}{k}$ the series will be convergent, and for all values of x beyond those limits it will be divergent.

Ex. $\frac{a}{1}x + \frac{a(a-1)}{1.2}x^2 + \frac{a(a-1)(a-2)}{1.2.3}x^3 + \&c.$

Here $\frac{a_{n+1}}{a_n} = \frac{a-n}{n+1} = -1$, when n is infinite;

therefore the series is convergent or divergent, according as x lies between $+1$ and -1, or without those limits.

Ex. $$\frac{a}{1} + \frac{a^2}{2} + \frac{a^3}{3} + \&c.,$$

$$u_n = \frac{a^n}{n},$$

$$\therefore\ \frac{u_{n+1}}{u_n} = \frac{na}{n+1} = a, \text{ when } n \text{ is infinite;}$$

therefore the series is convergent or divergent, according as $a <$ or > 1.

Ex. $$\frac{1}{1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots + \frac{1}{\lfloor n},$$

$$\frac{u_{n+1}}{u_n} = \frac{1}{n+1} = 0, \text{ when } n \text{ is infinite,}$$

therefore the series converges.

113. The series $u_1 + u_2 + \dots + u_n + \dots$ is convergent, or will become so, if the inferior limit of $\dfrac{\log u_n}{\log \frac{1}{n}}$ be > 1, when n is infinite; and divergent in the contrary case.

Let k denote the inferior limit of $\dfrac{\log u_n}{\log \frac{1}{n}}$ when n is infinite, and first, suppose $k > 1$; also let a be any number between k and 1, so that $k > a > 1$. Then when n is indefinitely increased, $\dfrac{\log u_n}{\log \frac{1}{n}}$ or its equal $\dfrac{\log \frac{1}{u_n}}{\log n}$ cannot approach indefinitely near to k without finally becoming constantly greater than a. Therefore it will be possible to take for n so large a value, that for that and all superior values we may constantly have

$$\log \frac{1}{u_n} > a \log n > \log n^a, \quad \text{or } \frac{1}{u_n} > n^a, \quad \text{or } u_n < \frac{1}{n^a}.$$

Consequently the proposed series will finish by always having its terms less than the corresponding terms of the converging series (Ex. Art. 112).

$$\frac{1}{1^a}+\frac{1}{2^a}+\frac{1}{3^a}+\ldots+\frac{1}{n^a}+\frac{1}{(n+1)^a}+\&c.$$

and therefore will itself end by being, *à fortiori*, convergent.

Similarly, if $k<1$, it may be shewn that the proposed series will finish by being divergent.

114. The series $u_0+u_1+u_2+u_3+u_4+u_5+\&c.$ (1), each term of which is less than the immediately preceding term,

and the series $u_0+2u_1+4u_3+8u_7+16u_{15}+\&c.$ (2), are convergent or divergent at the same time.

Suppose series (1) to be convergent and that its sum $=s$, then

$$u_0=u_0,$$

$$2u_1=2u_1,$$

$$4u_3<2u_2+2u_3,$$

$$8u_7<2u_4+2u_5+2u_6+2u_7$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore\; u_0+2u_1+4u_3+8u_7+16u_{15}+\&c.$$

$$<u_0+2(u_1+u_2+u_3+\&c.)<2s-u_0,$$

consequently series (2) is convergent. Next, suppose series (1) divergent, then

$$u_0=u_0,$$

$$2u_1>u_1+u_2,$$

$$4u_3>u_3+u_4+u_5+u_6$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\therefore\ u_0 + 2u_1 + 4u_3 + 8u_7 + \&c. > u_0 + u_1 + u_2 + u_3 + \&c.$

and is therefore, divergent.

Ex. Let series (1) be

$$\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \&c. \qquad (3),$$

then series (2) is

$$\frac{1}{1^m} + \frac{1}{2^{m-1}} + \frac{1}{4^{m-1}} + \frac{1}{8^{m-1}} + \&c.$$

a geometric progression convergent when $m > 1$, and divergent in the contrary case; consequently series (3) will be convergent if $m > 1$, and divergent if $m =$ or < 1.

115. The series $u_1 - u_2 + u_3 - u_4 + \&c.$, is convergent, if the numerical value of the terms decreases without limit.

For, by writing it in the forms

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - \&c.,$$

$$u_1 - u_2 + (u_3 - u_4) + \&c.;$$

we see that it is $> u_1 - u_2$ and $< u_1$, and therefore is convergent.

Thus, the sum of the series

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c.,$$

lies between 1 and $\frac{1}{2}$; also the series

$$\frac{1}{1^m} - \frac{1}{2^m} + \frac{1}{3^m} - \frac{1}{4^m} + \&c.,$$

is convergent for all positive values of m.

116. It is also shewn in the work from which these tests of convergency are taken, that two converging series all whose terms are positive, will by their addition or multiplication produce new converging series, whose sums result from the addition or multiplication of the sums of the former.

117. An approximate value of Σu_x will be obtained by taking the aggregate of the converging terms only, in the series for Σu_x involving $\int_x u_x$, u_x, and the differential coefficients of u_x; and the error will be less than the last of the convergent, or the first of the divergent terms.

We have by Art. 63, (omitting the index of u_x in the second member,)

$$\Sigma u_x = C + \int_x u - \tfrac{1}{2} u + \frac{B_1}{1.2} d_x u - \frac{B_3}{\lfloor 4} d_x^3 u + \dots$$

$$+ (-1)^{n-1} \frac{B_{2n-1}}{\lfloor 2n} d_x^{2n-1} u + (-1)^n 2R_n, \quad (1)$$

where $$2R_n = \frac{B_{2n+1}}{\lfloor 2n+2} d_x^{2n+1} u - \frac{B_{2n+3}}{\lfloor 2n+4} d_x^{2n+3} u + \text{\&c.};$$

or since in general, by Art. 67,

$$\frac{B_{2n-1}}{\lfloor 2n} = 2\left\{\frac{1}{(2\pi)^{2n}} + \frac{1}{(4\pi)^{2n}} + \frac{1}{(6\pi)^{2n}} + \text{\&c.}\right\};$$

$$R_n = \left\{\frac{1}{(2\pi)^{2n+2}} + \frac{1}{(4\pi)^{2n+2}} + \frac{1}{(6\pi)^{2n+2}} + \dots\right\} d_x^{2n+1} u$$

$$- \left\{\frac{1}{(2\pi)^{2n+4}} + \frac{1}{(4\pi)^{2n+4}} + \frac{1}{(6\pi)^{2n+4}} + \dots\right\} d_x^{2n+3} u$$

$$+ \left\{\frac{1}{(2\pi)^{2n+6}} + \frac{1}{(4\pi)^{2n+6}} + \frac{1}{(6\pi)^{2n+6}} + \dots\right\} d_x^{2n+5} u$$

$$- \text{\&c.};$$

or, adding the terms vertically, and calling the resulting series v_1, v_2, &c.,

$$R_n = v_1 + v_2 + v_3 + \ldots + v_m + \ldots$$

$$\text{where } v_m = \frac{d_x^{2n+1}u}{(2m\pi)^{2n+2}} - \frac{d_x^{2n+3}u}{(2m\pi)^{2n+4}} + \frac{d_x^{2n+5}u}{(2m\pi)^{2n+6}} - \&c.;$$

$$\therefore\ d_x^2 v_m = -(2m\pi)^2\left\{v_m - \frac{d_x^{2n+1}u}{(2m\pi)^{2n+2}}\right\},$$

$$\text{or } d_x^2 v_m + (2m\pi)^2 v_m = \frac{d_x^{2n+1}u}{(2m\pi)^{2n}},$$

and integrating this by the met od of parameters, as in Art. 72,

$$v_m = -\frac{1}{(2m\pi)^{2n+1}}\int_x(\sin 2m\pi x\,.\,d_x^{2n+1}u),$$

since, x being an integer, the term multiplied by $\sin 2m\pi x$ disappears, and $\cos 2m\pi x = 1$; and the arbitrary constant is unnecessary, being already introduced in equation (1);

$$\therefore\ R_n = -\int_x\left\{\frac{\sin 2x\pi}{(2\pi)^{2n+1}} + \frac{\sin 4x\pi}{(4\pi)^{2n+1}} + \frac{\sin 6x\pi}{(6\pi)^{2n+1}} + \&c.\right\} d_x^{2n+1}u;$$

therefore, numerically,

$$2R_n < -2\int_x\left\{\frac{1}{(2\pi)^{2n}} + \frac{1}{(4\pi)^{2n}} + \frac{1}{(6\pi)^{2n}} + \ldots\right\} d_x^{2n+1}u,$$

$$< -\frac{B_{2n-1}}{\underline{|2n}}\, d_x^{2n}u,$$

which last quantity lies between

$$\frac{B_{2n-1}}{\underline{|2n}}\, d_x^{2n-1}u, \quad\text{and}\quad \frac{B_{2n+1}}{\underline{|2n+2}}\, d_x^{2n+1}u;$$

if these be the last of the convergent and the first of the divergent terms respectively, of the series for Σu_x. Consequently, the sum of all the diverging terms in the series for Σu_x, is less than the last of the convergent or the first of the divergent terms.

Interpolation.

118. When a series of values of a quantity is obtained either by observation, or laborious calculation, it is of great importance to be able to insert other values between them, such as would have resulted from a similar observation or calculation, without the labour of performing these. This is the object of Interpolation; and in this the Calculus of Finite Differences finds one of its chief uses. More strictly, Interpolation of Series is the inserting among the terms of a given series, new terms subject to the same law as the first. In doing this, the terms of the series are considered as particular values of the function which expresses its general term, corresponding to a given regular succession of indices; and it is the business of Interpolation to discover that general term; or at least to assign such a function of the index as shall represent the given series of values, and, approximately, all intermediate values. The problem thus requiring us to assign the analytical expression of a function from a limited number of its numerical values, is plainly indeterminate; it is the same as to form the equation to a curve which shall pass through a limited number of points, whose abscissæ represent the values of the independent variable, and the ordinates those of the function, without giving the species of the curve; which, as is evident, may be done in an infinite variety of ways. But if the given terms are numerous, and near to each other, the expression for the general term, within the limits of the given quantities, may be found to a great degree of accuracy.

119. There are two principal cases to be considered; first, when the given values of $f(x)$, namely,

$$f(x_1),\quad f(x_1+h),\quad f(x_1+2h),\ldots\quad f\{x_1+(n-1)h\},\ \text{or}$$
$$u_1,\quad u_2,\quad u_3,\ \ldots,\ u_n,$$

as we shall write them, correspond to values of the independent variable, x_1, x_1+h, x_1+2h, &c., $x_1+(n-1)h$, in arithmetical progression. And secondly, when the given values $f(x_1)$, $f(x_2)$, ... $f(x_n)$, or u_1, u_2, ... u_n, correspond to values x_1, x_2, ... x_n, of the independent variable, not obeying any assigned law.

120. Having given u_1, u_2, u_3, &c. u_n, n values of a function $f(x)$, corresponding to the n values of the independent variable x_1, x_1+h, ... $x_1+(n-1)h$, to find an expression for any intermediate value $f(x_1+k)$.

$$\text{Since } f(x+nh) = f(x) + n\Delta f(x) + \frac{n(n-1)}{1.2}\Delta^2 f(x) + \text{\&c.}$$

(Art. 23) changing x into x_1, and then replacing nh by k, we get

$$f(x_1+k) = u_1 + \frac{k}{h}\Delta u_1 + \frac{k(k-h)}{1.2h^2}\Delta^2 u_1 + \text{\&c.}$$

$$+ \frac{k(k-h)\ldots\{k-(n-2)h\}}{\underline{|n-1}\,.\,h^{n-1}}\Delta^{n-1}u_1,$$

in which, since

$$\left.\begin{aligned}
\Delta u_1 &= u_2 - u_1, \\
\Delta^2 u_1 &= u_3 - 2u_2 + u_1, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\Delta^{n-1} u_1 &= u_n - (n-1)u_{n-1} + \frac{(n-1)(n-2)}{1.2}u_{n-2} - \text{\&c.} \ldots \pm u_1,
\end{aligned}\right\}(2).$$

if we make $k=0$, h, $2h$, &c., $(n-1)h$, the second member assumes the n values u_1, u_2, ... u_n; and not only this, but if we assume for k any value whatever between 0 and $(n-1)h$, we shall obtain the value of the function corresponding to that value of the independent variable.

121. In applying the above formula, the simplest mode is not to calculate Δu_1, $\Delta^2 u_1$, &c. by equations (2), but by continued subtraction of the given terms; that is, we must

write down the series of given values $u_1, u_2, \ldots u_n$, and subtract each from the succeeding one; next subtract each of these differences from the succeeding difference; then perform the same operation upon the new differences; and so on, till the process terminates; the first terms of these series of differences are the values of $\Delta u_1, \Delta^2 u_1, \ldots \Delta^{n-1} u_1$. Unless the terms of the given series by continued subtraction lead to a constant difference, the expression for $f(x_1 + k)$ will have as many terms as the given series has.

Ex. 1. Having given the values of sin 30°, sin 31°, sin 32°, sin 33°, to find $\sin(30^0 + k')$, k being between 0 and 180′. Here

$u_1 = .5$	Δ		
$u_2 = .5150381$	150381	Δ^2	
$u_3 = .5299193$	148812	-1569	Δ^3
$u_4 = .5446390$	147197	-1615	-46;

$\therefore \quad \Delta u_1 = 0{\cdot}0150381, \quad \Delta^2 u_1 = -0{\cdot}0001569, \quad \Delta^3 u_1 = -0{\cdot}0000046,$

$$\therefore \quad \sin(30^0 + k') = .5 + \frac{k}{60}\Delta u_1 + \frac{k(k-60)}{2.60^2}\Delta^2 u_1 + \frac{k(k-60)(k-120)}{2.3.60^3}\Delta^3 u_1.$$

If $k = 20$, it will be found that $\sin 30^0.20' = .5050299$, which is too large only by a unit in the seventh place of decimals.

Ex. 2. Having given

$\log 3.14 = 0.496929$, $\log 3.15 = 0.498310$, $\log 3.16 = 0.499687$,

$\log 3.17 = 0.501059$; shew that $\log 3.14159 = 0.497149$.

Ex. 3. Having given u_1, u_2, u_3, u_4, four right ascensions (declinations, longitudes, &c.) of the moon at intervals of 12 hours, to find its value t hours after the time corresponding to the second value. Here $h = 12$, $k = 12 + t$, and if the required right ascension $= u_2 + \delta$, then

$$u_2+\delta=u_1+\left(1+\frac{t}{12}\right)\Delta u_1+\frac{(t+12)t}{2.12^2}\Delta^2 u_1+\frac{(t+12)t(t-12)}{2.3.12^3}\Delta^3 u_1,$$

$$\text{or } \delta=\left(\Delta^1+\frac{1}{2}\Delta^2-\frac{1}{6}\Delta^3\right)\frac{t}{12}+\frac{1}{2}\Delta^2\left(\frac{t}{12}\right)^2+\frac{1}{6}\Delta^3\left(\frac{t}{12}\right)^3,$$

when developed in powers of $\frac{t}{12}$.

122. Between every two consecutive terms of a given series, to interpolate any number of equidistant terms.

Let u_1, u_2, u_3, &c., u_n be the given series, and let $m-1$ be the number of equidistant terms to be inserted between every two consecutive terms; then the new series will be

$$u_1,\ u_{\frac{m+1}{m}},\ u_{\frac{m+2}{m}},\ \dots\ u_{\frac{2m-1}{m}},\ u_2,\ u_{\frac{2m+1}{m}},\ \text{\&c.}$$

If therefore v_{r+1} denote the $r+1^{\text{th}}$ term of this series, we have

$$v_{r+1}=u_{\frac{m+r}{m}}=f\left(1+\frac{r}{m}\right),$$

$$\text{or}\quad v_{r+1}=u_1+\frac{r}{m}\Delta u_1+\frac{r(r-m)}{1.2m^2}\Delta^2 u_1+\text{\&c.}$$

Hence, taking r from 1 to $m-1$, we get the terms inserted between u_1 and u_2; next taking r from $m+1$ to $2m-1$, we get the terms between u_2 and u_3; and so on. The differences Δu_1, $\Delta^2 u_1$, &c., are to be computed by continued subtraction as in Art. (121); and the series for v_{r+1} will have as many terms as the proposed series has, unless those terms by continued subtraction lead to a constant difference.

Ex. To insert three equidistant terms between every two consecutive ones of the series 1, 7, 15, 28, 49, &c. Here $m=4$, and

Δ,	6,	8,	13,	21,
Δ^2	2,	5,	8,	
Δ^3	3,	3,		

$$\therefore\ v_{r+1} = 1 + \frac{6r}{4} + \frac{r(r-4)}{16} + \frac{r(r-4)(r-8)}{128} = \frac{128 + 192r - 4r^2 + r^3}{128};$$

and the series is $1,\ \frac{317}{128},\ \frac{504}{128},\ \frac{695}{128},\ 7,\ \frac{1113}{128}$, &c.

123. The formula of Art. 120 may be presented under a different form by changing k into $x - x_1$, which gives

$$f(x) = u_1 + \frac{x - x_1}{h}\Delta u_1 + \frac{(x - x_1)(x - x_1 - h)}{1.2.h^2}\Delta^2 u_1 + \&c.$$

$$+ \frac{(x - x_1)(x - x_1 - h) \dots \{x - x_1 - (n-2)h\}}{\underline{|n-1}.h^{n-1}}\Delta^{n-1} u_1;$$

where $f(x)$ is a function of x which, as x assumes the n values x_1, $x_1 + h$, &c., $x_1 + (n-1)h$, successively assumes the corresponding values $u_1, u_2, \dots u_n$; and for any other value of x within, or not far beyond, the limits x_1 and $x_1 + (n-1)h$, it gives the value of the corresponding interpolated term. If we put $h = 1$, the formula is adapted to the case where the increment of the principal variable is unity.

124. In any series of consecutive equidistant values of a function, where one is deficient to insert that one.

Let u_1, u_2, u_3, &c., u_n be the values of the function corresponding to the values $x_1, x_1 + h, \dots x_1 + (n-1)h$ for x. Then assuming that $\Delta^{n-1} u_1 = 0$, or that the $(n-2)^{\text{th}}$ differences are constant, which will almost always be the case in tabulated results, we have

$$\Delta^{n-1} u_1 = u_n - (n-1)u_{n-1} + \frac{(n-1)(n-2)}{1.2}u_{n-2} - \&c. \pm u_1 = 0,$$

an equation of the first degree from which any one of the values as u_r may be found, if the rest be known. Having thus completed the system of values, we may interpolate any intermediate term $f(x_1 + k)$ by the method of Art. 120. If two values out of n are deficient, then we must suppose

$$\Delta^{n-2}u_1 = 0, \quad \Delta^{n-2}u_2 = 0, \text{ or}$$

$$u_{n-1} - (n-2)\,u_{n-2} + \frac{(n-2)\,(n-3)}{1\,.\,2}\,u_{n-3} - \&c. \pm u_1 = 0,$$

$$u_n - (n-2)\,u_{n-1} + \frac{(n-2)\,(n-3)}{1\,.\,2}\,u_{n-2} - \&c. \pm u_2 = 0;$$

which equations will suffice to determine any two of the values in terms of the rest. In the same way any number of deficient terms may be inserted.

Ex. 1. Given the cube roots of 121, 122, 124, 125, to find that of 123.

$$u_1 = 4.946088, \quad u_2 = 4.959675, \quad u_4 = 4.986631, \quad u_5 = 5,$$

$$\Delta^4 u_1 = u_5 - 4u_4 + 6u_3 - 4u_2 + u_1 = 0;$$

$$\therefore \quad u_3 = \tfrac{1}{6}\left\{4\,(u_2 + u_4) - u_1 - u_5\right\} = 4.973190.$$

Ex. 2. Given u_1, u_2, u_5, u_6, to find u_3, u_4.

$$\Delta^4 u_1 = u_5 - 4u_4 + 6u_3 - 4u_2 + u_1 = 0.$$

$$\Delta^4 u_2 = u_6 - 4u_5 + 6u_4 - 4u_3 + u_2 = 0;$$

$$\therefore \quad u_3 = \frac{1}{10}\,(-3u_1 + 10u_2 + 5u_5 - 2u_6),$$

$$u_4 = \frac{1}{10}\,(-2u_1 + 5u_2 + 10u_5 - 3u_6).$$

125. We next come to the case where the given values

$$f(x_1),\quad f(x_2),\ \&c.,\quad f(x_n),\quad \text{or } u_1,\quad u_2, \ldots, u_n,$$

correspond to values x_1, $x_2, \ldots x_n$, not obeying any assigned law; and it is required to determine a rational integral function of $n-1$ dimensions, $f(x)$, which shall assume the n given values u_1, $u_2, \ldots u_n$, when for x the values x_1, x_2, x_3, &c., x_n are successively substituted.

Since $f(x)$ is of $(n-1)$ dimensions, we may assume

$$\frac{f(x)}{(x-x_1)(x-x_2)\ldots(x-x_n)} = \frac{C_1}{x-x_1} + \frac{C_2}{x-x_2} + \ldots + \frac{C_n}{x-x_n};$$

$$\therefore\ f(x) = C_1(x-x_2)(x-x_3)\ldots(x-x_n)$$

$$+ C_2(x-x_1)(x-x_3)\ldots(x-x_n) + \ldots + C_n(x-x_1)(x-x_2)\ldots(x-x_{n-1}).$$

Now make $x = x_1$, x_2, &c., x_n, successively; and observing that the corresponding values of the first member are u_1, $u_2, \ldots u_n$, we get

$$u_1 = C_1(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n),$$

$$u_2 = C_2(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_n = C_n(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1});$$

$$\therefore\ f(x) = u_1\frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}$$

$$+ u_2\frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} + \&c.$$

$$+ u_n\frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})},$$

which is Lagrange's Theorem for interpolation.

Ex. To find a function of x which, when $x = 1, 3, 6, 12$, shall assume the values $1, 7, 10, -8$.

$$f(x) = -\frac{(x-3)(x-6)(x-12)}{2.5.11} + 7.\frac{(x-1)(x-6)(x-12)}{2.3.9}$$

$$-\frac{(x-1)(x-3)(x-12)}{3.3} - 4.\frac{(x-1)(x-3)(x-6)}{11.9.3}.$$

126. To determine the maximum or minimum value of a function, from three of its values near its maximum or minimum, and the three corresponding values of the independent variable.

If u_1, u_2, u_3 be the given values of u, and x_1, x_2, x_3 those of x, we have

$$u = u_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + u_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + u_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}.$$

Hence, putting $d_x u = 0$, we find

$$u_1(x_2-x_3)(2x-x_2-x_3) + u_2(x_3-x_1)(2x-x_1-x_3)$$

$$+ u_3(x_1-x_2)(2x-x_1-x_2) = 0;$$

$$\therefore\ x = \frac{u_1(x_2^2-x_3^2) - u_2(x_1^2-x_3^2) + u_3(x_1^2-x_2^2)}{2u_1(x_2-x_3) - 2u_2(x_1-x_3) + 2u_3(x_1-x_2)},$$

the value of x at which u is a maximum or minimum. This formula is useful in various Astronomical problems, as for instance, to determine the meridian altitude of a heavenly body, when an observation exactly on the meridian cannot be obtained.

Fig. 1.

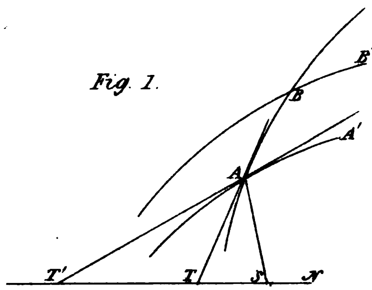


Fig. 2.

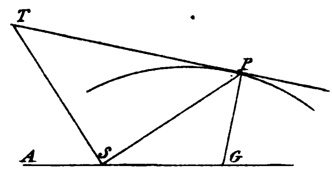


Fig. 3.

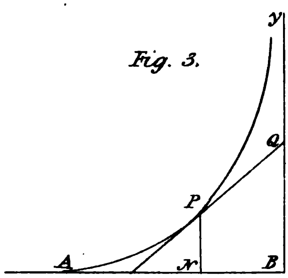


Fig. 4.

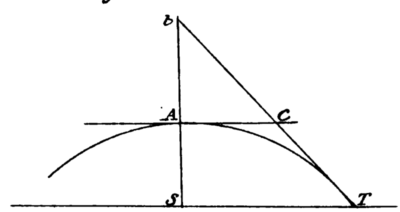
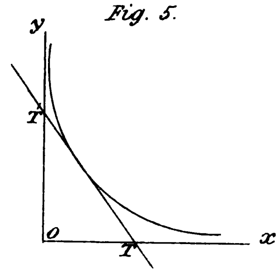


Fig. 5.



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